Chapter 4

Theory of lift

1.Kutta–Joukowski Theory

1. 1.Circulation and Vortex

Definition of Circulation

Circulation Γ is a macroscopic measure of rotation in a flow field. For a closed curve C, it is defined as the line integral of velocity around the curve:

$$\boldsymbol{\Gamma = \oint_C \mathbf{V} \cdot d\mathbf{s}} \quad \textbf{(VI.1)}$$

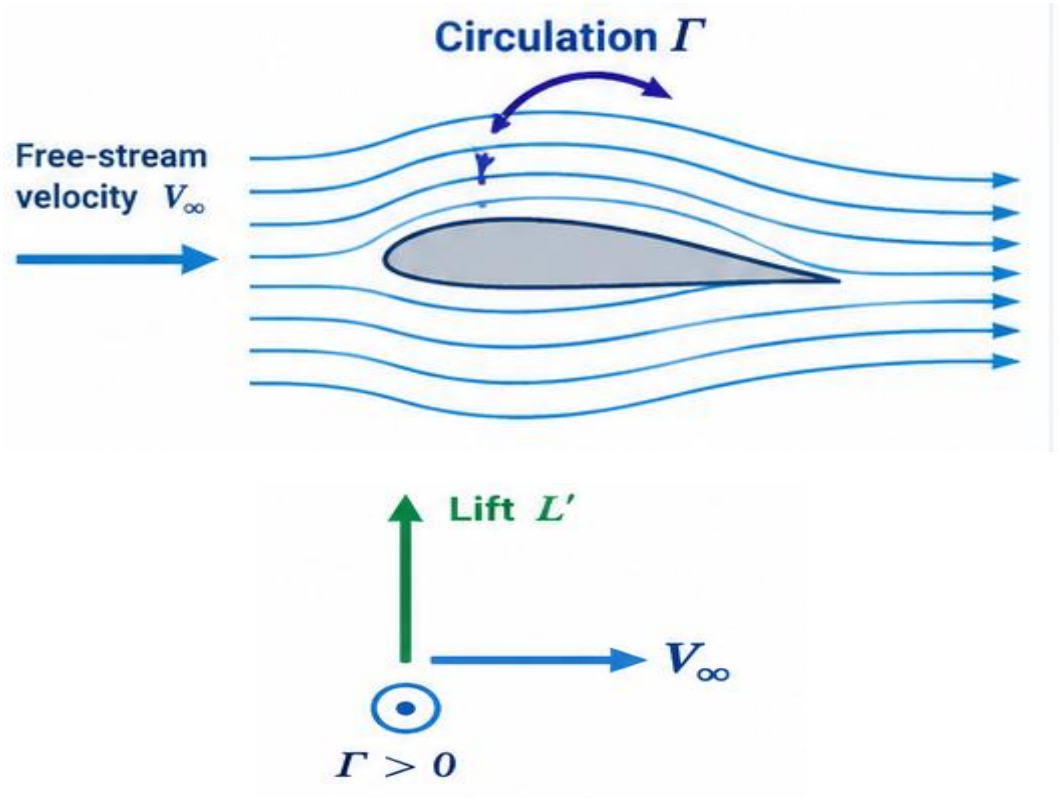


Figure 1.Circulation.

In Anderson's Fundamentals of Aerodynamics , circulation is related to vorticity ξ=∇ ×V through Stokes' theorem:

$$\boldsymbol{\Gamma = \iint_S (\nabla \times \mathbf{V}) \cdot d\mathbf{S}} \quad \textbf{(VI.2)}$$

This is Stokes' theorem, which relates the circulation around a closed curve (*C*) to the curl of the velocity field over the surface (*S*) bounded by (*C*).

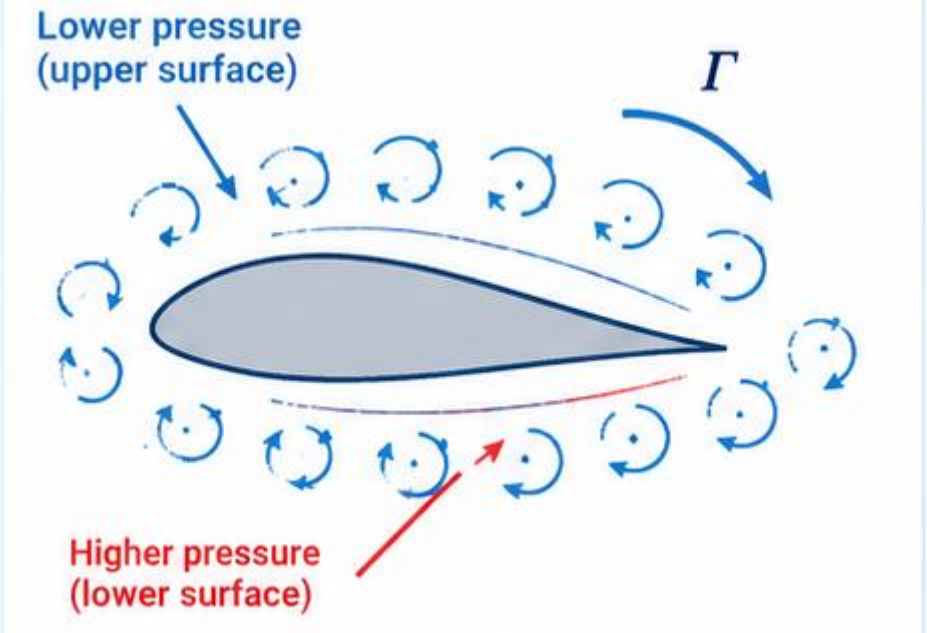


Figure 2:Pressure.

2.Vorticity and Rotational Flow

Vorticity is twice the angular velocity of a fluid element:

$$\boldsymbol{\xi = 2\omega = \nabla \times V} \quad \textbf{(VI.3)}$$

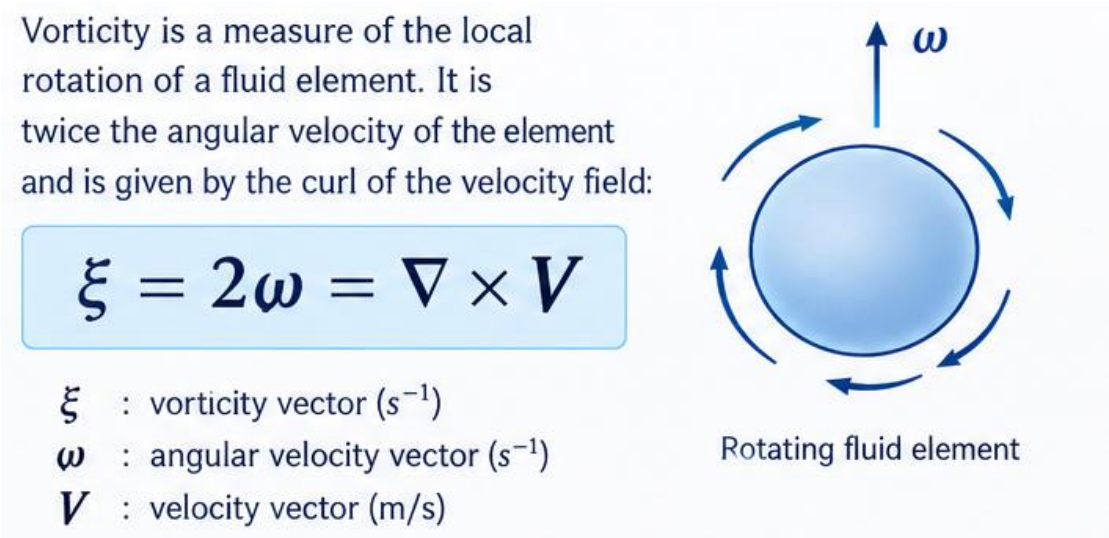


Figure 3.Vorticity.

- If □×**V≠0** everywhere: **rotational flow**
- **If** □×**V=0** everywhere: **irrotational flow** (potential flow)

For a two-dimensional flow, the irrotationality condition is:

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \text{ (VI.4)}$$

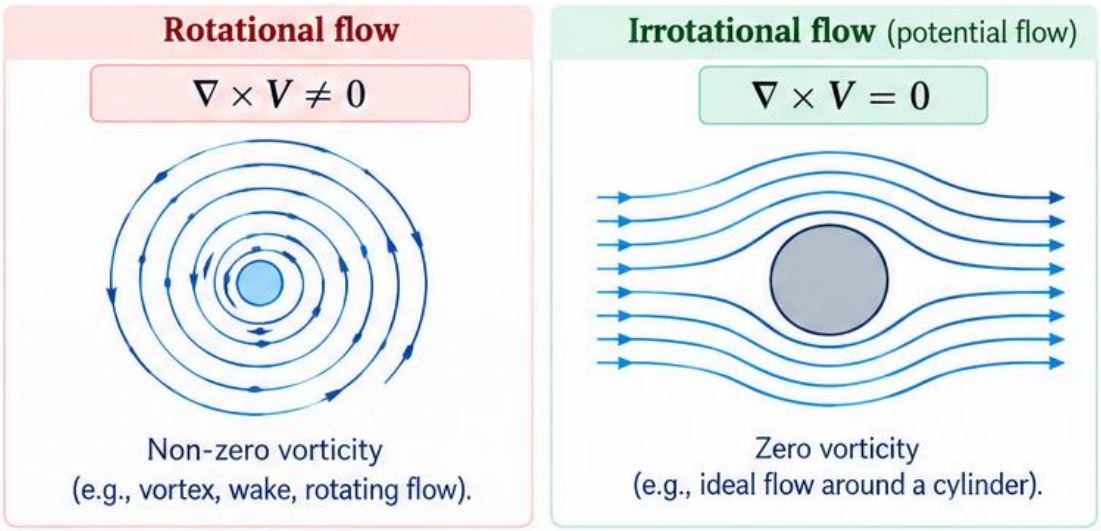


Figure 4.Rotational an Irrotational flow.

Note: Here, u is the velocity component in the x-direction, and v is the velocity component in the y-direction. The vorticity component perpendicular to the two-dimensional flow is ($\xi_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$).

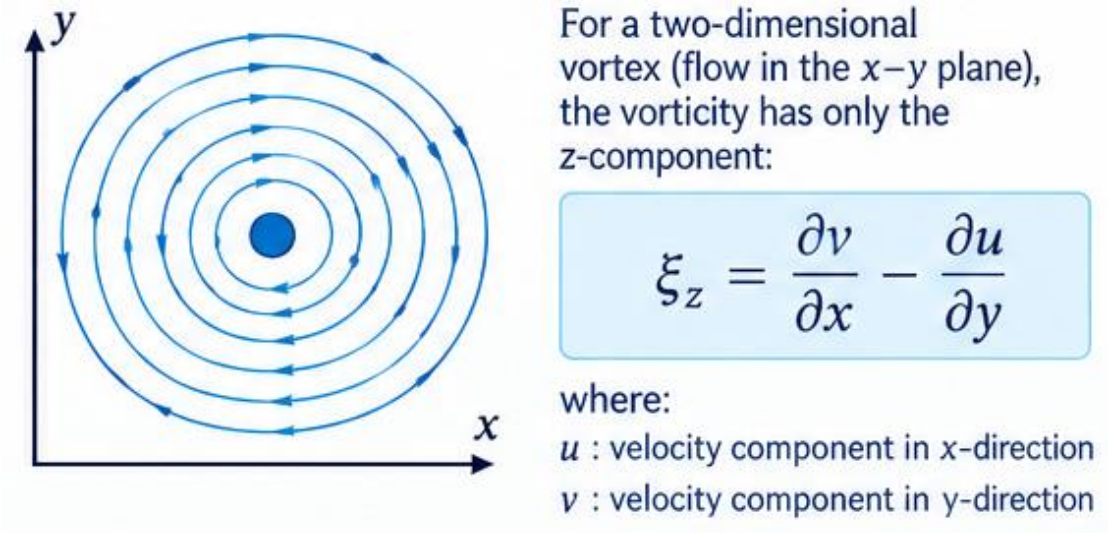
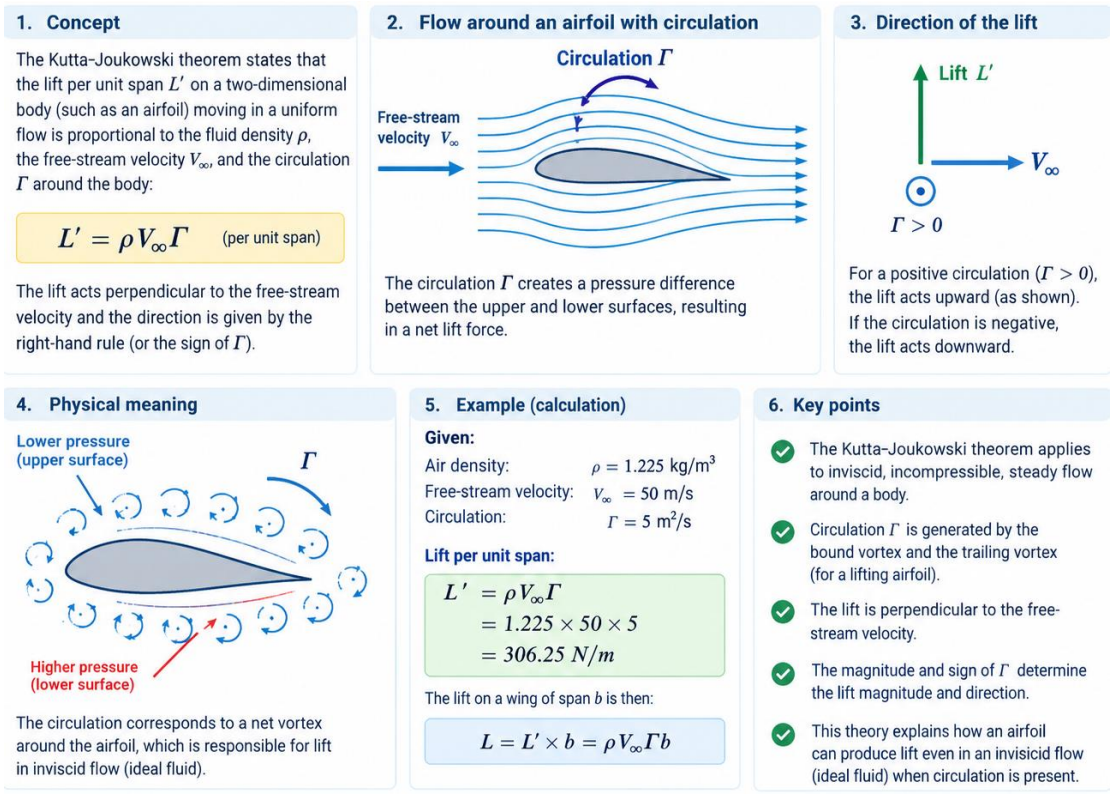


Figure 5.Two-dimensional vortex.



2.1.The Vortex Sheet and Kutta Condition

In thin airfoil theory , the airfoil surface is replaced by a **vortex sheet** of variable strength γ(s). Across the sheet, there is a tangential velocity discontinuity:

$$\boldsymbol{\gamma = u_1 - u_2} \quad \textbf{(VI.5)}$$

The **Kutta condition** requires that the flow leave the trailing edge smoothly:

$$\boldsymbol{\gamma(TE)=0}$$

This condition is what fixes the circulation around the airfoil — nature chooses the unique value of Γ that satisfies smooth flow at the trailing edge. Kelvin's circulation theorem then explains the generation of this circulation through the starting vortex.

3. Wing Theory (2D and 3D)

3.1.2D: Thin Airfoil Theory

The fundamental equation of thin airfoil theory is:

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)}{x-\xi}\,d\xi = V_\infty\left(\alpha - \frac{dz}{dx}\right) \quad \textbf{(VI.6)}$$

Symmetric airfoil results:

- Lift coefficient: $c_l = 2\pi\alpha$
- Lift slope: $\frac{dc_l}{d\alpha} = 2\pi$ **rad**$^{-1}$

Center of pressure and aerodynamic center at quarter chord

$$c_{m,c/4} = 0$$

Cambered airfoil results:

$$c_l = 2\pi\left[\alpha + \frac{1}{\pi}\int_0^\pi \frac{dz}{dx}(\cos\theta_0 - 1)\,d\theta_0\right] \quad \textbf{(VI.7)}$$

- Lift slope still 2π
- Aerodynamic center at quarter chord; center of pressure varies with *cl*

3.2.3D: Prandtl's Lifting-Line Theory

For a finite wing , the wing is replaced by a lifting line with circulation Γ(y) varying along the span, plus a trailing vortex sheet.

The downwash at spanwise station y_0 is:

$$w(y_0) = -\frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{d\Gamma}{dy}\frac{dy}{y_0 - y} \quad \textbf{(VI.8)}$$

The fundamental equation of lifting-line theory

$$\alpha(y_0) = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_{L=0}(y_0) + \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{d\Gamma}{dy}\frac{dy}{y_0 - y} \quad \textbf{(VI.9)}$$

Elliptical lift distribution (minimum induced drag):

$$\Gamma(\theta) = \Gamma_0 \sin\theta \quad \textbf{(VI.10)}$$

Results for elliptic distribution:

Downwash is constant:

$$w = -\frac{\Gamma_0}{2b} \quad \textbf{(VI.11)}$$

Induced angle:

$$\alpha_i = \frac{C_L}{\pi AR} \quad \textbf{(VI.12)}$$

Induced drag:

$$C_{D,i} = \frac{C_L^2}{\pi AR} \quad \textbf{(VI.13)}$$

General lift distribution via Fourier series:

$$\Gamma(\theta) = 2bV_\infty \sum_{n=1}^{N} A_n \sin(n\theta) \quad \textbf{(VI.14)}$$

with:

$$C_L = \pi AR\, A_1$$

$$C_{D,i} = \pi AR \sum_{n=1}^{N} n\, A_n^2 = \frac{C_L^2}{\pi AR}(1+\delta) \quad \textbf{(VI.15)}$$

Span efficiency factor:

$$e = \frac{1}{1+\delta} = (1+\delta)^{-1} \quad \textbf{(VI.16)}$$

The lift slope for a finite wing is reduced from the airfoil value:

$$a = \frac{a_0}{1+\frac{a_0}{\pi AR}(1+\tau)} \quad \textbf{(VI.17)}$$

For high-speed subsonic flow, compressibility corrections apply (Prandtl–Glauert)

$$C_p = \frac{C_{p,0}}{\sqrt{1-M_\infty^2}} \quad \textbf{(VI.18)}$$

4. Lift Distribution

4.1.Spanwise Lift Distribution

Elliptical planform: produces elliptical lift distribution and minimum induced drag — the theoretical optimum

Rectangular wing: lift drops sharply near tips; higher induced drag (δ≈0.05-0.1)

Tapered wing: with taper ratio λ=ct/cr≈0.4-0.5, approaches elliptical efficiency with easier manufacturing

The lift per unit span is:

$$L'(y) = \rho_\infty V_\infty \Gamma(y) \quad \textbf{(VI.19)}$$

Total lift:

$$L = \rho_\infty V_\infty \int_{-b/2}^{b/2} \Gamma(y)\, dy \ \textbf{(VI.20)}$$

4.2.Chordwise (2D) Distribution

On the airfoil section itself, the pressure distribution (and hence local lift) is governed by profile shape and incidence:

The upper (suction) surface carries a strong negative Cp peak near the leading edge
Increasing thickness reduces the depth of the suction peak
increasing camber shifts loading and lowers the leading-edge peak
The stagnation point sits on the lower surface near the leading edge at positive incidence

4.3.Delta Wings (Special 3D Case)

For low-aspect-ratio delta wings, classical lifting-line theory fails. The dominant feature is the **leading-edge vortex pair** above the wing, which creates strong suction on the upper surface near the leading edges. This:

- Enhances lift at high angles of attack
- Allows the wing to stall at much higher α (e.g., 35°)

Makes classical lifting-line theory inappropriate for such planforms — vortex-lattice or lifting-surface methods are needed

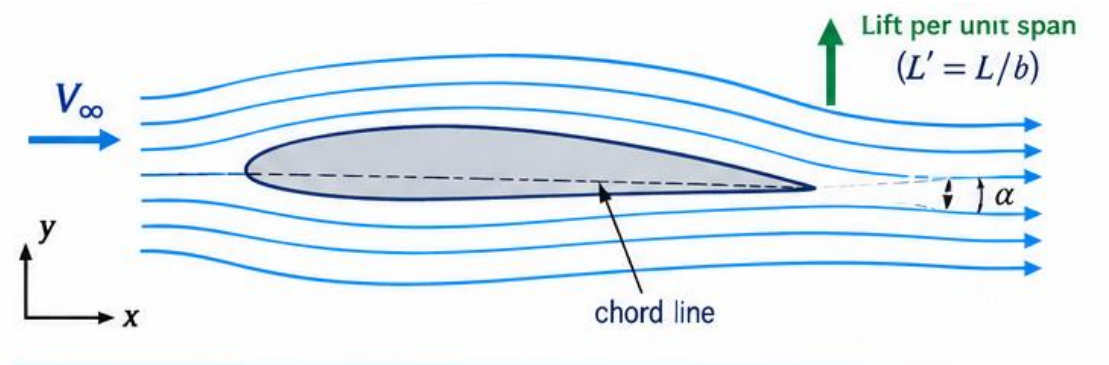
4.4.Reynolds Analogy (Connection to Skin Friction)

From boundary-layer theory , Reynolds analogy relates heat transfer to skin friction:

$$C_{H,c_f} = \frac{1}{2} Pr^{-2/3} \quad \textbf{(VI.21)}$$

and the recovery factor for a flat plate is

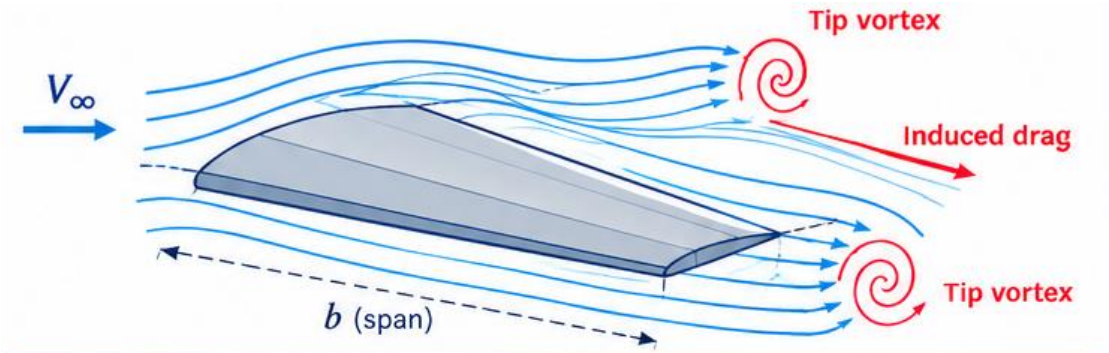
$$r = Pr^{1/3} \textbf{ (VI.22)}$$



Lift per unit span (2D):

$$L' = \rho V_\infty \Gamma$$

ρ : fluid density (kg/m³)
V_∞ : freestream velocity (m/s)
Γ : circulation around the airfoil (m²/s)



Total lift (finite wing):

$$L = \rho V_\infty \Gamma b$$

L : total lift (N)
b : wing span (m)
Γ : average circulation (m²/s)

Aspect ratio (AR):

$$AR = \frac{b^2}{S}$$

S : wing planform area (m²)
Higher AR → lower induced drag
Lower AR → higher induced drag

Exercise 1 — Circulation around a Closed Curve

The velocity field of a two-dimensional flow is u = −2y and v = 2x, where u and v are in m/s and x, y are in meters. Calculate the circulation around a circular path of radius R = 2 m, centered at the origin and traversed counterclockwise.

Solution

Circulation:$\Gamma = \oint CV \cdot ds = \oint C(udx + vdy)$.
Parameterize the circle: $x = R\cos\theta, y = R\sin\theta;\ dx = -R\sin\theta\, d\theta$ and $dy = R\cos\theta\, d\theta$
. Substitute $u = -2Rsin\theta and v = 2Rcos\theta$
:$\Gamma = \int_0^2\pi [2R^2 \sin^2\theta + 2R^2\cos^2\theta]\, d\theta = 4\pi R^2$.
For R = 2 m: $\Gamma = 16\pi \approx 50.27\ m^2/s$.
Interpretation: positive circulation corresponds to a counterclockwise sense under the usual sign convention.

Exercise 2 — Vorticity and Flow Classification

Consider the velocity field u = 3x + 2y and v = x − 3y. (1) Calculate the vorticity component ξz. (2) Classify the flow as rotational or irrotational. (3) Calculate circulation around a counterclockwise circular path of radius 1 m centered at the origin.

Solution

$$\xi z = \partial v/\partial x - \partial u/\partial y = 1 - 2 = -1s^{-1}.$$

Since ξz ≠ 0, the flow is rotational.
By Stokes' theorem, $\Gamma = \iint S\xi z dS = \xi z\pi R^2 = -\pi \approx -3.142 m^2/s.$

The negative sign indicates clockwise circulation under the counterclockwise-positive convention.

Exercise 3 — Lift Using the Kutta–Joukowski Theorem

An airfoil is placed in uniform airflow with air density ρ∞ = 1.225 kg/m³, freestream velocity V∞ = 40 m/s, and circulation magnitude |Γ| = 2.5 m²/s. Calculate the lift per unit span.

Solution

The Kutta–Joukowski theorem gives $L' = \rho\infty V\infty|\Gamma|$
Therefore, $L' = 1.225 \times 40 \times 2.5 = 122.5 N/m$
.Answer: L′ = 122.5 N/m.

Exercise 4 — Lift Coefficient of a Symmetric Airfoil

A symmetric airfoil operates at an angle of attack of 5°. Assume thin airfoil theory is valid. (1) Calculate Cl. (2) Calculate lift per unit span for ρ∞ = 1.225 kg/m³, V∞ = 50 m/s, and chord c = 1.2 m.

Solution

Convert angle: α = 5° × π/180 = 0.08727 rad.
Thin airfoil theory: Cl = 2πα = 2π(0.08727) ≈ 0.5483.
Lift per unit span: L′ = ½ ρ∞ V∞² c Cl.
L′ = ½ × 1.225 × 50² × 1.2 × 0.5483 ≈ 1,005 N/m.

Exercise 5— Lift and Induced Drag of a Finite Wing

A finite wing has $S = 20\ m^2$, $b = 10\ m$, $C_L = 0.6$, $V_\infty = 50\ m/s$, $\rho_\infty = 1.225\ kg/m^3$, and span efficiency $e = 0.9$. Calculate (1) aspect ratio, (2) total lift, (3) induced drag coefficient, and (4) induced drag.

Solution

Aspect ratio: $AR = b^2/S = 10^2/20 = 5$.
Lift: $L = \frac{1}{2}\rho_\infty V_\infty^2 S C_L = \frac{1}{2} \times 1.225 \times 50^2 \times 20 \times 0.6 = 18{,}375\ N$.
Induced drag coefficient: $C_{D,i} = C_L^2/(\pi AR\ e) = 0.6^2/[\pi \times 5 \times 0.9] \approx 0.02546$.
Induced drag: $D_i = \frac{1}{2}\rho_\infty V_\infty^2 S C_{D,i} \approx 779.7\ N$.

Exercise 6— Elliptical Lift Distribution

A wing has span $b = 12\ m$ and elliptical circulation distribution $\Gamma(y) = \Gamma_0\sqrt{[1 - (2y/b)^2]}$, where $\Gamma_0 = 4\ m^2/s$. Calculate circulation at the center, at $y = 3\ m$, and at the wingtip; then calculate total lift for $\rho_\infty = 1.225\ kg/m^3$ and $V_\infty = 50\ m/s$.

Solution

At $y = 0$: $\Gamma(0) = \Gamma_0 = 4\ m^2/s$.
At $y = 3\ m$: $\Gamma(3) = 4\sqrt{[1 - (6/12)^2]} = 4\sqrt{0.75} \approx 3.464\ m^2/s$.
At the tip $y = b/2 = 6\ m$: $\Gamma(6) = 0$.
For an elliptical distribution, $\int\Gamma(y)dy$ from $-b/2$ to $b/2 = \pi b\Gamma_0/4$.
Total lift: $L = \rho_\infty V_\infty \pi b \Gamma_0/4 = 1.225 \times 50 \times \pi \times 12 \times 4/4 \approx 2{,}309\ N$.

Exercise 7 — Lift Slope of a Finite Wing

A finite wing has two-dimensional airfoil lift slope $a_0 = 2\pi\ rad^{-1}$, aspect ratio $AR = 8$, and correction factor $\tau = 0.05$. Using $a = a_0/[1 + a_0(1 + \tau)/(\pi AR)]$, calculate the finite-wing lift slope, C_L at $\alpha = 6°$ (zero-lift angle = 0), and compare with the two-dimensional airfoil coefficient.

Solution

$a = 2\pi/[1 + (2\pi/(\pi \times 8))(1.05)] = 6.2832/1.2625 \approx 4.977\ rad^{-1}$.
$\alpha = 6\pi/180 = 0.10472\ rad$.
Finite wing: $C_L = a\alpha = 4.977 \times 0.10472 \approx 0.521$.
Two-dimensional airfoil: $C_l = 2\pi\alpha = 2\pi \times 0.10472 \approx 0.658$.
The finite wing has a lower lift coefficient at the same angle because of downwash and induced angle of attack.

Exercise 8 — Prandtl–Glauert Compressibility Correction

An airfoil has incompressible pressure coefficient $C_{p,0} = -0.8$ at a point on its upper surface. The freestream Mach number is $M_\infty = 0.6$. Calculate the corrected pressure coefficient using the Prandtl–Glauert correction.

Solution

$C_p = C_{p,0}/\sqrt{(1 - M_\infty^2)} = -0.8/\sqrt{(1 - 0.6^2)} = -0.8/0.8 = -1.0$.
Answer: $C_p = -1.0$. This linearized correction has limitations near transonic conditions and when local supersonic regions occur.

Exercise 10 — Reynolds Analogy and Heat Transfer

For a flat plate, the skin-friction coefficient is Cf = 0.004 and the Prandtl number is Pr = 0.72. Using CH,cf = ½ Cf Pr^(−2/3) and recovery factor r = Pr^(1/3), calculate CH,cf and r.

Solution

- CH,cf = ½ × 0.004 × (0.72)^(−2/3) ≈ 0.00249.
- r = (0.72)^(1/3) ≈ 0.896.