Chapter 2

Flow Around Bodies

1. Flow Around a Cylinder and a Sphere

These two geometries are classical cases for the study of external flows (flow around an immersed body). Their behavior is governed by a fundamental dimensionless number:

- Reynolds Number (Re)

$$\boldsymbol{Re = \frac{\rho V D}{\mu} = \frac{V D}{\nu}} \quad \textbf{(II.1)}$$

Where:

- ρ: **fluid density**
- V: **upstream flow velocity**
- D: **diameter of the obstacle** (cylinder or sphere)
- μ: **dynamic viscosity**
- ν: **kinematic viscosity**

The **Reynolds number** represents the ratio of **inertial forces to viscous forces**. It plays a key role in determining the flow structure and flow regime.

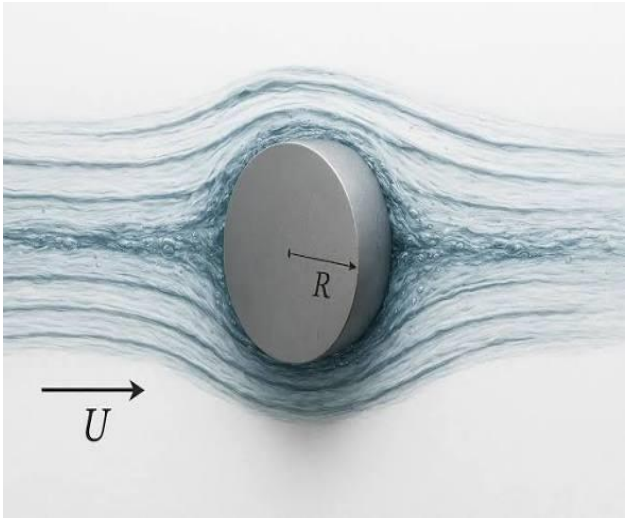


Figure 1.flow around an immersed body.

The boundary layer is a thin region of fluid adjacent to a solid surfac**e**, such as an aircraft wing, a flat plate, a cylinder, or a sphere.

1.1 Ideal Flow (Inviscid, Incompressible) Around a Cylinder

In aerodynamics, the study of ideal flow around a circular cylinder is a classic problem. Because the flow is assumed to be **inviscid** (no friction) and **incompressible** (constant density), we can use the principle of superposition to build this flow field mathematically.

A Uniform Flow: Velocity V∞ moving in the x-direction.

A Doublet: A singularity of strength κ placed at the origin, oriented to face upstream (against the uniform flow).

When these are combined, the resulting stream function is:

$$\boldsymbol{\psi = V_\infty r \sin\theta \left(1 - \frac{R^2}{r^2}\right)} \quad \textbf{(II.2)}$$

where R is the radius of the cylinder, defined by $\boldsymbol{R^2 = \frac{\kappa}{2\pi V_\infty}}$

The streamline where ψ=0 forms a perfect circle of radius R, which represents the solid surface of the cylinder.

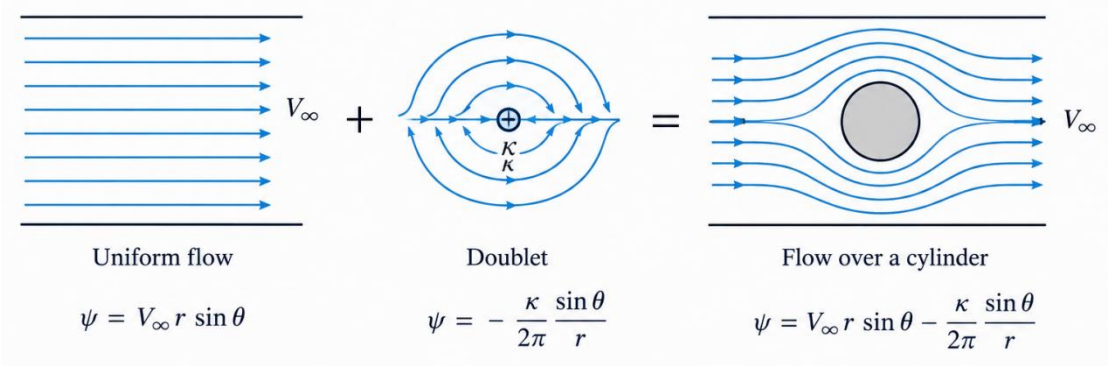


Figure 2.Ideal Flow Around a Cylinder.

Nonlifting cylinder: A uniform flow combined with a doublet creates the flow around a circular cylinder of radius R. The stream function is:

$$\boldsymbol{\psi = (V_\infty r \sin\theta)\left(1 - \frac{R^2}{r^2}\right)} \quad \textbf{(II.3)}$$

The velocity at the surface is:

$$\boldsymbol{V_\theta = -2V_\infty \sin\theta} \quad \textbf{(II.4)}$$

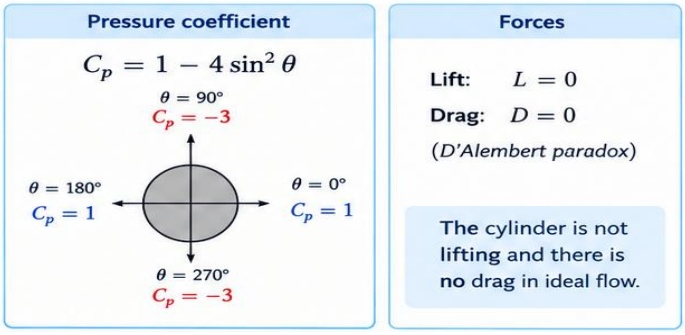


Figure 3.The pressure coefficient.

The pressure coefficient is:

$$\boldsymbol{C_p = 1 - 4\sin^2\theta} \quad \textbf{(II.5)}$$

The lift and drag are zero (d'Alembert's paradox).

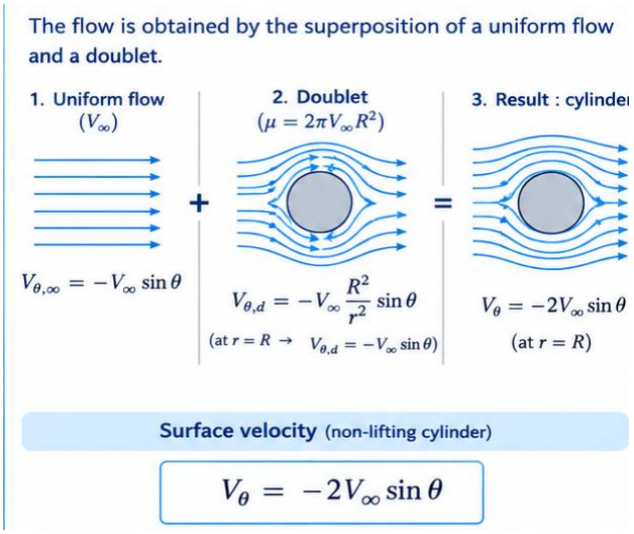


Figure 4.Nonlifting cylinder.

Cylindre portant: En ajoutant un tourbillon de circulation Γ, on obtient un écoulement portant. La vitesse de surface devient

$$\boldsymbol{V_\theta = -2V_\infty \sin\theta - \frac{\Gamma}{2\pi R}} \quad \textbf{(II.6)}$$

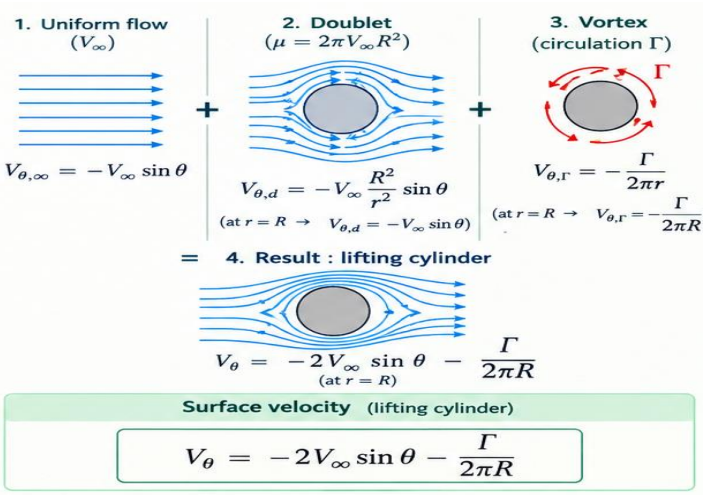


Figure 5.Cylindre portant

The lift per unit span is given by the Kutta-Joukowski theorem:

$$L' = \rho_\infty V_\infty \Gamma$$

La traînée reste nulle.

Key idea

- The final velocity is obtained by the principle of **superposition**: we add the velocities of the individual flows.
- **Non-lifting cylinder**: uniform flow + doublet → $V_\theta = -2V_\infty \sin\theta$
- **Lifting cylinder**: uniform flow + doublet + vortex → $V_\theta = -2V_\infty \sin\theta - \frac{\Gamma}{2\pi R)}$

Table 1.Variation of the Drag Coefficient with the Reynolds Number

Reynolds Number (Re)	Flow Regime	Drag Coefficient (CD)	Characteristics
0–4	Stokes	Very high	Attached, symmetrical flow
4–40	Stable	≈ 1.5	Two stable vortices downstream
40–300	Unstable	≈ 1.2	Kármán vortex street
300–3×10^5	Subcritical	≈ 1.2	Laminar separation (≈ 80°)
≈ 3×10^5	Critical	Drops to ≈ 0.3	Turbulent transition, delayed separation
> 3×10^6	Supercritical	≈ 0.6	Turbulent separation (≈ 120°)

1.2. Ideal Flow Around a Sphere

- A **uniform flow** combined with a **three-dimensional doublet** produces the flow around a sphere. The velocity at the sphere surface is:

$$\boldsymbol{V_\theta = -\frac{3}{2} V_\infty \sin\theta} \quad \textbf{(II.7)}$$

The pressure coefficient is:

$$\boldsymbol{C_p = 1 - \frac{9}{4}\sin^2\theta} \quad \textbf{(II.8)}$$

Three-dimensional relief effect: The maximum velocity on a sphere, 1.5 V_∞, is lower than that on a cylinder, $2V_\infty$, for the same V_∞. Therefore, the minimum pressure is less negative, which **reduces the severity of the flow acceleration**.

1.3.Real (Viscous) Flow Around a Cylinder and a Sphere

- The **drag is not zero** because of **friction and flow separation**. The drag coefficient C_D depends strongly on the **Reynolds number**:

$$\boldsymbol{Re = \frac{\rho_\infty V_\infty d}{\mu_\infty}} \quad \textbf{(II.9)}$$

- **Cylinder:** For Re<1, corresponding to **Stokes flow**, C_D is very high and decreases as Re increases.

For: $10^3 < Re < 3 \times 10^5$

the drag coefficient is approximately:$C_D \approx 1$

At approximately:$Re \approx 3 \times 10^5$

there is a **sharp decrease in** C_D associated with the transition from a laminar to a turbulent boundary layer. Then, C_D increases again to approximately:

$$C_D \approx 0.6$$

$$Re = 10^7$$

Sphere: The drag is lower than that of the cylinder due to the three-dimensional relief effect. (C_D≈0.4) for (Re<3×10^5), then decreases to (C_D≈0.1) for (Re>3×10^5).

Table 2.Comparison: Cylinder vs. Sphere

Quantity	Cylinder	Sphere
$Vmax/V\infty$	2	1.5
Cp,min	−3	−1.25
$C_D(Re \approx 10^5)$	$\approx$ 1.2	$\approx$ 0.4
$C_D(Re > 3 \times\times 10^5)$	$\approx$ 0.3	$\approx$ 0.1

2. Boundary Layer Concept (Prandtl's Theory)

Definition: Introduced by Ludwig Prandtl in 1904, the boundary layer is a thin region of flow adjacent to a solid surface where frictional (viscous) effects are significant. Outside this layer, the flow can be considered **inviscid (non-viscous)**.

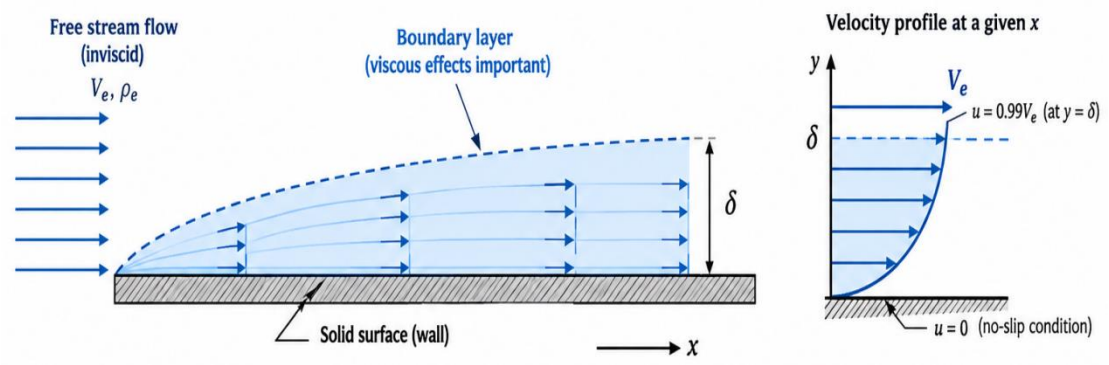


Figure 6.Prandtl's Theory

2.1.Fundamental Assumptions of Prandtl (1904):

1. $\delta \square c$:The boundary-layer thickness is much smaller than the characteristic dimension of the body.
2. $Re \square 1$:The Reynolds number is high.

No-Slip Condition:

- u=v=w=0 at the wall

Thermal Boundary Condition:

- T=Tw at the wall

2.2.Fundamental Properties

No-slip condition: The fluid velocity at the wall is zero (V = 0). The velocity gradually increases away from the wall until it reaches **0.99Ve** at the boundary of the boundary layer, whose thickness is denoted by **δ**.

Displacement thickness (δ*): This represents the displacement of the external streamlines caused by the mass-flow deficit within the boundary layer.

$$\delta^* = \int_0^\infty \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy \quad \textbf{(II.10)}$$

Momentum thickness (θ): This represents the momentum deficit caused by the presence of the boundary layer.

$$\theta = \int_0^\infty \frac{\rho u}{\rho_e u_e}\left(1 - \frac{u}{u_e}\right) dy \quad \textbf{(II.11)}$$

These definitions are particularly important for **quantifying the effects of the boundary layer on the external flow**.

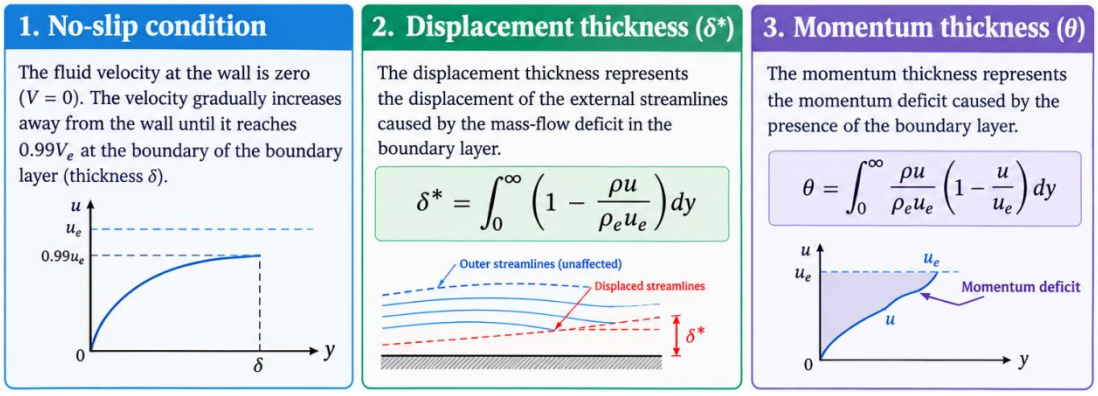


Figure 7.Fundamental Properties

3.Boundary Layer Equations

These are the Navier–Stokes equations simplified using an order-of-magnitude analysis, valid for $\delta \ll c$ and large Re. The momentum equation in the y-direction shows that the pressure remains constant across the boundary layer:

$$\frac{\partial p}{\partial y} = 0 \qquad \textbf{(II.12)}$$

Simplified Equations for Steady Two-Dimensional Flow:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \ \textbf{(II.13)}$$

x-Momentum Equation:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dp_e}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \quad \textbf{(II.14)}$$

y-Momentum Equation:

$$\frac{\partial p}{\partial y} = 0 \quad \Rightarrow \quad p = p_e(x) \ \textbf{(II.15)}$$

Energy Equation:

$$\frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + u \frac{dp_e}{dx} + \mu \left(\frac{\partial u}{\partial y}\right)^2 \textbf{(II.16)}$$

Boundary Conditions:

- At the wall $((y=0)): u = v = 0, \qquad T = T_w$
- Outside the boundary layer $y \to \infty$

$$u \to u_e, \qquad T \to T_e$$

3.1.Blasius Solution for Flow over a Flat Plate

Assumptions:

- Incompressible flow
- Flat plate at zero incidence
- $\frac{dp_e}{dx} = 0$

Blasius Equation:

$$2f''' + ff'' = 0 \quad \textbf{(II.17)}$$

Where

$$f'(\eta) = \frac{u}{u_e} \quad \text{and} \quad \eta = y\sqrt{\frac{u_e}{\nu x}} \quad \textbf{(II.18)}$$

Numerical Solution:

$$f''(0) = 0.332$$

Table 3.Solution de Blasius pour une plaque plane

Quantity	Expression
Local skin-friction coefficient	$c_f = \frac{0.664}{\sqrt{Re_x}}$
Average skin-friction coefficient	$C_f = \frac{1.328}{\sqrt{Re_c}}$
Boundary-layer thickness	$\delta = \frac{5.0x}{\sqrt{Re_x}}$
Displacement thickness	$\delta^* = \frac{1.72x}{\sqrt{Re_x}}$
Momentum thickness	$\theta = \frac{0.664x}{\sqrt{Re_x}}$

These are the standard **laminar flat-plate Blasius results**, with $Re_x = \frac{u_e x}{\nu}$ and $Re_c = \frac{u_e c}{\nu}$

4. Flow Separation and Vortex Formation

4.1.Separation Mechanism

1. An **adverse pressure gradient** (pressure increasing in the direction of flow, dp/dx > 0) slows down the fluid within the boundary layer.
2. The fluid particles near the wall, already slowed down by friction, no longer have enough kinetic energy to overcome the adverse pressure gradient.
3. Their velocity decreases to zero and then becomes negative, indicating **reverse flow**. This location is known as the **separation point**.
4. The flow **separates from the surface**, forming a **recirculation region (vortices)** and a **wake** downstream.

4.2.Formation of Vortices

- **Wake vortices:** In the recirculation region, the flow becomes disordered and forms vortices.
- **Kármán vortex street:** For a cylinder, when Re > 40, vortices are alternately shed from each side of the cylinder, forming a **vortex street**.
- **Wingtip vortices:** For a finite wing, flow separation at the wingtips generates **wingtip vortices**, which induce **induced drag**.
- **Consequences:** Flow separation significantly modifies the pressure distribution over the rear of the body, considerably increasing **pressure drag (form drag)** and reducing **lift (stall)**.

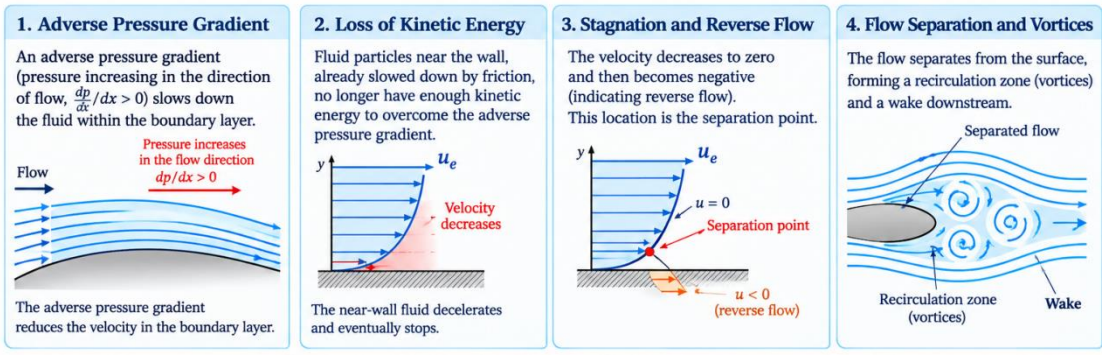


Figure 7.Flow Separation and Vortex Formation

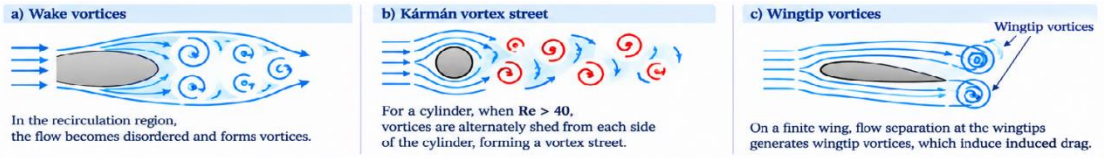


Figure 8.example of Vortex Formation

5. Influence of Shape on Drag

5.1.Concept of streamlined vs. bluff bodies:

- **Bluff body:** A thick shape (e.g., a cylinder or a flat plate perpendicular to the flow). The flow separates early, creating a large wake, and **pressure drag** is dominant. The drag coefficient CD is high (e.g., $C_D = 2$ for a flat plate and $C_D = 1.2$ for a cylinder).

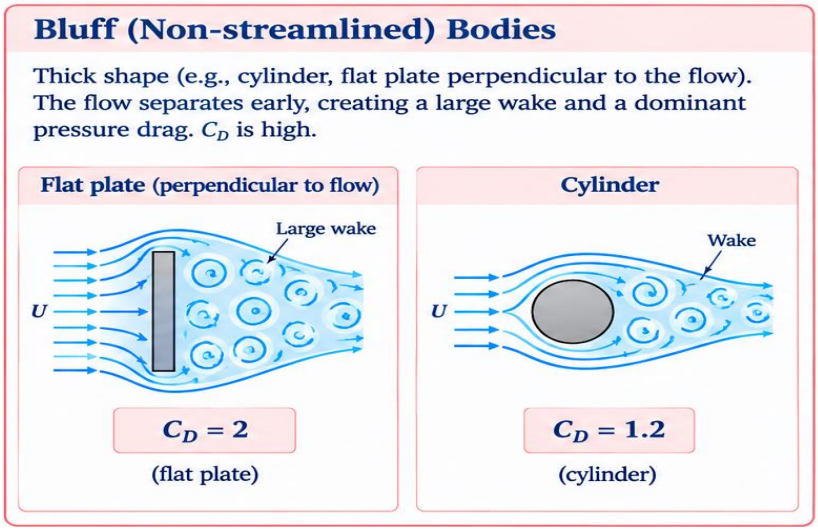


Figure 9.Bluff body.

- **Streamlined body:** An elongated and tapered shape (e.g., an airfoil or a streamlined body of revolution). The flow remains attached for a longer distance, reducing the wake and pressure drag. The drag is mainly due to **friction (skin-friction drag)**. The drag coefficient CD is low (e.g., CD = 0.12 for a streamlined body).

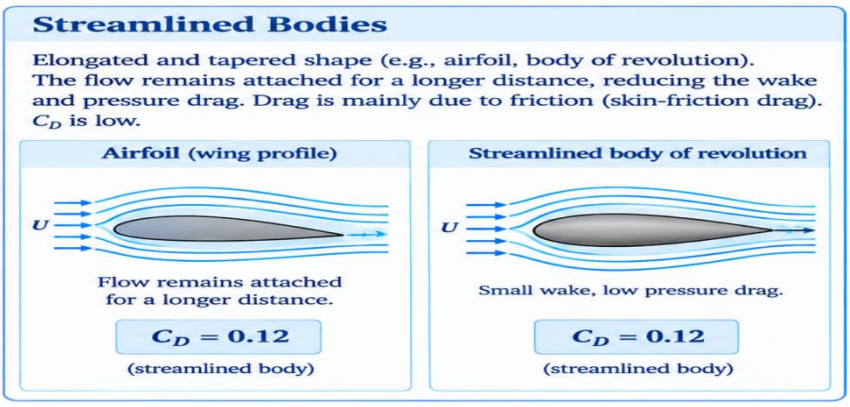


Figure 10.Streamlined body.

5.2.Effect of Shape on Drag

- A **streamlined body** can have a total drag similar to that of a much smaller **bluff body**. For example, the drag of a streamlined body with a thickness `d` is equal to that of a cylinder with a diameter of `0.1d`.
- **Optimization:** To minimize drag, flow separation should be delayed by giving the body an **aerodynamic (streamlined) shape** and promoting **turbulent flow**, which is more resistant to separation than laminar flow. This principle is illustrated by the **dimples on a golf ball**.

Exercise 1: Reynolds Number and Flow Regime Around a Cylinder

A cylinder of diameter D=0.1 m is placed in an airflow at standard sea level conditions (ρ=1.225 kg/m^3, μ=1.789×10^{-5} kg/(m· s)).

The upstream velocity is V=15 m/s.

1. Calculate the Reynolds number.
2. Determine the flow regime.
3. Estimate the drag coefficient C_D.

Solution:

1. Reynolds number:

$$Re=\frac{\rho VD}{\mu}$$

$$Re=\frac{(1.225)(15)(0.1)}{1.789\times10^{-5}}$$

$$Re=\frac{1.8375}{1.789\times10^{-5}}$$

$$Re\approx1.027\times10^{5}$$

$$\boxed{Re\approx1.03\times10^{5}}$$

2. Flow regime:

From Table 1:

$$300<Re<3\times10^5 \quad \rightarrow \quad Subcriticalregime$$

-

Laminar separation occurs at approximately 80∘ 80∘ from the stagnation point.

-

3. Drag coefficient:

From Table 1, for the subcritical regime:

$$C_D \approx 1.2$$

Exercise 2: Ideal Flow Around a Cylinder — Pressure Coefficient

Problem:

Consider ideal (inviscid, incompressible) flow around a cylinder of radius R = 0.5 m. The freestream velocity is V_∞ = 20 m/s, and the freestream pressure is $p_\infty = 1.01 \times 10^5$ Pa. The fluid is air with $\rho = 1.225$ kg/m³.

1. Calculate the velocity at the top of the cylinder ($\theta = 90°$).
2. Calculate the pressure coefficient Cp at this point.
3. Calculate the actual pressure p at this point.

Solution:

1. Velocity at the surface:

From Eq. (II.4):

$$V_\theta = -2V_\infty \sin\theta$$

At $\theta = 90°$, $\sin 90° = 1$:

$$V_\theta = -2(20)(1) = -40 \text{ m/s}$$

The magnitude is:

$$V = 40 \text{ m/s}$$

2. Pressure coefficient:

From Eq. (II.5):

$$Cp = 1 - 4\sin^2\theta$$

At $\theta = 90°$:

$$Cp = 1 - 4(1)^2 = 1 - 4 = -3$$

$$Cp = -3$$

3. Actual pressure:

$Cp = (p - p\infty) / (\frac{1}{2}\rho V\infty^2)$

$p = p\infty + Cp \cdot (\frac{1}{2}\rho V\infty^2)$

$\frac{1}{2}\rho V\infty^2 = \frac{1}{2}(1.225)(20)^2 = 245$ Pa

$p = 1.01 \times 10^5 + (-3)(245)$

$p = 101000 - 735 = 100265$ Pa

$p \approx 1.003 \times 10^5$ Pa

Exercise 3: Ideal Flow Around a Sphere — Three-Dimensional Relief Effect

Problem:

A sphere of diameter D = 0.2 m is placed in the same airflow as Exercise 2 ($V\infty$ = 20 m/s, ρ = 1.225 kg/m³).

1. Calculate the maximum velocity on the sphere surface.
2. Calculate the minimum pressure coefficient Cp,min.
3. Compare with the cylinder results from Exercise 2.

Solution:

1. Maximum velocity on the sphere:

From Eq. (II.7):

$$V\theta = -(3/2)V\infty \sin\theta$$

At $\theta = 90°$:

$$Vmax = (3/2)(20) = 30 \text{ m/s}$$

$$Vmax = 30 \text{ m/s}$$

2. Minimum pressure coefficient:

From Eq. (II.8):

$$Cp = 1 - (9/4)\sin^2\theta$$

At $\theta = 90°$:

$$C_{P,min} = 1 - 9/4 = 1 - 2.25 = -1.25$$

$$C_{P,min} = -1.25$$

3. Comparison:

Quantity	Cylinder	Sphere
Vmax	40 m/s	30 m/s
Cp,min	−3	−1.25

The sphere experiences a lower maximum velocity and a less negative minimum pressure coefficient than the cylinder. This is the three-dimensional relief effect: the flow has an extra dimension to move around the sphere, reducing the severity of the acceleration and the suction peak.

Exercise 4: Real Flow — Drag Coefficient Comparison

Problem:

Using Table 2, compare the drag coefficients of a cylinder and a sphere at two Reynolds numbers:

1. $Re = 10^5$
2. $Re = 5 \times 10^5$

The frontal area of both bodies is $S = 0.01\ m^2$, and the dynamic pressure is $q\infty = 500\ Pa$.

Calculate the drag force for each case.

Solution:

From Table 2:

Quantity	Cylinder	Sphere
C_D at $Re \approx 10^5$	≈ 1.2	≈ 0.4
C_D at $Re > 3 \times 10^5$	≈ 0.3	≈ 0.1

Drag force formula:

$$D = C_D \cdot q\infty \cdot S$$

1. At $Re = 10^5$:

Cylinder:

$$D = (1.2)(500)(0.01) = 6N$$

$$D_{cylinder} = 6N$$

Sphere:

$D = (0.4)(500)(0.01) = 2\ N$

$$D_{sphere} = 2N$$

2. At $Re = 5 \times 10^5$:

Cylinder:

$$D = (0.3)(500)(0.01) = 1.5N$$

$$D_{cylinder} = 1.5N$$

Sphere:

$$D = (0.1)(500)(0.01) = 0.5N$$

$$D_{sphere} = 0.5N$$

Interpretation:

• The sphere has a lower drag coefficient than the cylinder for the values listed in Table 2.

• At high Reynolds number ($Re > 3 \times 10^5$), the listed drag coefficient decreases significantly for both bodies because transition to a turbulent boundary layer can delay separation and reduce pressure drag.

Exercise 5: Blasius Solution — Flat Plate Boundary Layer

Problem:

Air flows over a flat plate at $V\infty = 30$ m/s. The plate has a chord $c = 1$ m. Standard sea level conditions: $\rho = 1.225$ kg/m³, $\nu = 1.46 \times 10^{-5}$ m²/s.

Assuming laminar flow over the entire plate, calculate:

1. The Reynolds number at the trailing edge.
2. The boundary-layer thickness δ at the trailing edge.
3. The local skin-friction coefficient cf at the trailing edge.
4. The average skin-friction coefficient C_f.
5. The total friction drag per unit span.

Solution:

1. Reynolds number at the trailing edge:

$$Re_c = V\infty c/\nu = (30)(1)/(1.46\times10^{-5})$$

$$Re_c \approx 2.055\times10^{6}$$

$$Re_c \approx 2.06\times10^{6}$$

2. Boundary-layer thickness at the trailing edge:

For a laminar flat-plate boundary layer:

$$\delta = 5.0x/\sqrt{Re_x}$$

At x = c = 1 m:

$$\delta = 5.0(1) / \sqrt{(2.055 \times 10^{6})}$$

$$\delta \approx 3.49 \times 10^{-3} \text{ m} = 3.49 \text{ mm}$$

3. Local skin-friction coefficient at the trailing edge:

$$C_F = 0.664 / \sqrt{Re_X}$$

$$C_F = 0.664 / \sqrt{(2.055 \times 10^{6})}$$

$$C_F \approx 4.63\times10^{-4}$$

4. Average skin-friction coefficient:

$$C_F = 1.328/\sqrt{Re_c}$$

$$C_F = 1.328/\sqrt{(2.055\times10^{6})}$$

$$C_F \approx 9.26\times10^{-4}$$

5. Total friction drag per unit span:

$$D' = C_f \cdot (1/2)\rho V\infty^2 \cdot c$$

$$D' = (9.26 \times 10^{-4}) \cdot (1/2)(1.225)(30)^2 \cdot (1)$$

$$D' = (9.26 \times 10^{-4})(551.25)$$

$$D' \approx 0.510 \text{ N/m}$$

Exercise 6: Flow Separation — Adverse Pressure Gradient

Problem:

Consider the flow over a circular cylinder in the subcritical regime ($Re \approx 10^5$). The boundary

layer separates at approximately $\theta = 80°$ from the stagnation point.

1. Explain why separation occurs at this location.
2. What happens to the pressure distribution after separation?
3. How does this affect the drag coefficient?

Solution:

1. Why separation occurs:

On the front of the cylinder ($0° < \theta < 90°$), the flow accelerates and the pressure decreases (favorable pressure gradient, $dp/dx < 0$). This helps the boundary layer remain attached.

On the rear of the cylinder ($\theta > 90°$), the flow decelerates and the pressure increases (adverse pressure gradient, $dp/dx > 0$). The fluid particles near the wall, already slowed by friction, do not have enough kinetic energy to overcome this adverse pressure gradient. Their velocity drops to zero and then reverses, causing the boundary layer to separate. For the assumed subcritical case, separation is taken to occur at approximately $\theta = 80°$.

2. Pressure distribution after separation:

Before separation, the pressure distribution is closer to the ideal (inviscid) prediction. After separation, a low-pressure wake forms behind the cylinder, and the rear-surface pressure does not recover to the freestream value. The rear pressure is therefore much lower than the ideal attached-flow prediction.

3. Effect on drag coefficient:

The pressure on the front of the cylinder is higher than on the rear. This pressure imbalance creates a net force in the flow direction — the pressure drag (or form drag). In the subcritical regime, the tabulated value is approximately $C_D \approx 1.2$, which is much larger than the friction drag alone.

If the boundary layer becomes turbulent near $Re \approx 3 \times 10^5$, separation can be delayed to approximately $\theta \approx 120°$, the wake becomes narrower, and the tabulated drag coefficient can drop sharply to approximately $C_D \approx 0.3$.