Chapitre I

Introduction to Aerodynamics

1. Introduction

Aerodynamics is a branch of fluid dynamics that studies air flows and their effects on solid objects, such as an aircraft wing. The term “aerodynamics” is often used as a synonym for “gas dynamics”; the difference is that gas dynamics applies to the study of the motion of all gases and is not limited to air.

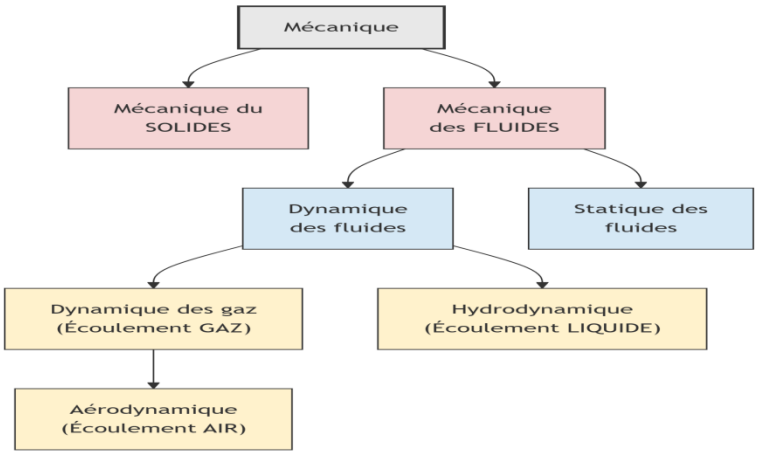
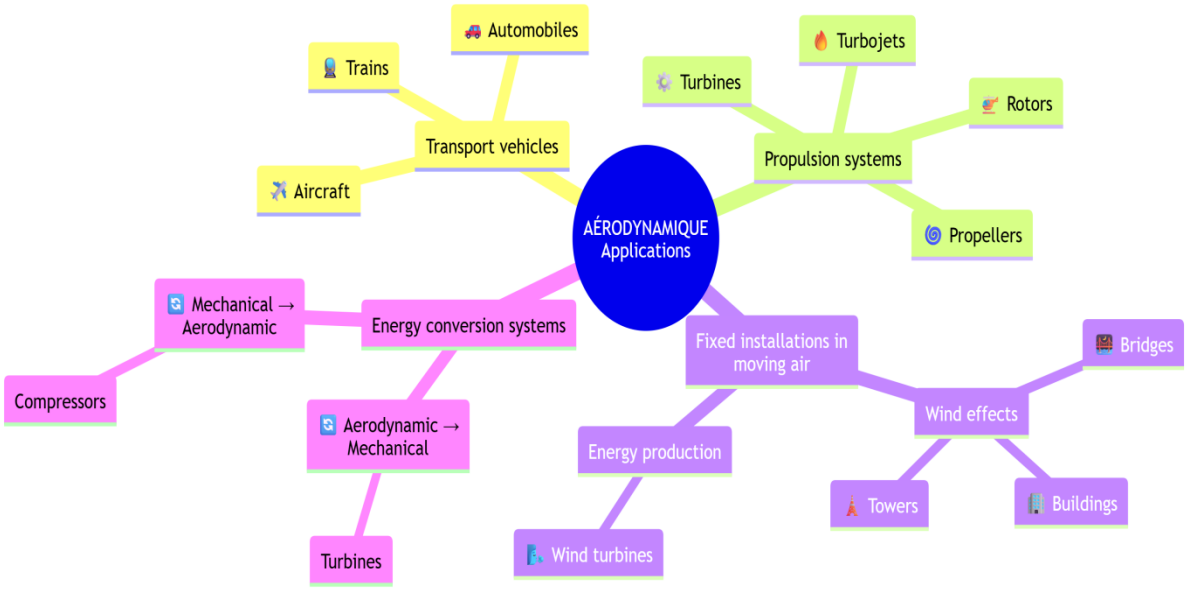


Figure 1. Classification of aerodynamics within classical mechanics.

Aerodynamics applies to the following systems:

- Transport vehicles (aircraft, automobiles, trains).
- Propulsion systems (propellers, rotors, turbines, turbojets).
- Fixed installations in moving air that are subjected to wind effects (buildings, towers, bridges), or intended for energy production (wind turbines).
- Systems converting aerodynamic energy into mechanical energy and vice versa (turbines, compressors).



2. Physical Properties

The physical properties of air are essential for understanding and analyzing its behavior in aerodynamics, whether for aircraft design, flow studies, or numerical simulation. This section recalls two important properties: density and viscosity.

2.1 Density

The density of a fluid is generally defined as the ratio of the mass of the fluid to its volume. Density plays a crucial role in calculating aerodynamic forces such as lift and drag. The higher the density, the greater the force exerted by the fluid on an object moving through it through dynamic pressure. Density is denoted by ρ (rho).

$$\rho = m / V \quad \text{(I.1)}$$

where m is the mass of the fluid [kg] and V is the fluid volume [m³]. Thus, the unit of density ρ is [kg/m³]. Liquids have an almost constant density and do not vary greatly with temperature and pressure, whereas the density of gases changes significantly with temperature and pressure.

In physics and engineering, an ideal gas is a theoretical gas model in which intermolecular forces are neglected. This ideal-gas model is a useful simplification that facilitates calculations and analysis. The ideal-gas law makes it possible to calculate the density of a gas from its pressure and temperature. Air can be treated as an ideal gas in many situations, particularly when pressures are not very high and temperatures are not very low.

$$P = \rho R T \quad \text{(I.2)}$$

where P is air pressure [Pa], T is air temperature [K], and R is the specific gas constant for air, R = 287 J kg^{-1} K^{-1}.

2.2 Viscosity

One of the most important parameters of a fluid is its viscosity. Viscosity determines the resistance of a fluid to flow and is a measure of its resistance to deformation at a given rate. The higher the viscosity, the less easily the fluid flows. Viscosity varies among fluids and no fluid has zero viscosity. Air has a low viscosity compared with water, and water has a low viscosity compared with honey. Viscosity may be sufficiently small to be neglected, particularly sufficiently far from the boundary layer of an airfoil.

2.2.1 No-Slip (Adherence) Condition

One consequence of fluid viscosity is that the fluid adheres to a solid surface when it is in contact with it. At the interface between a solid and a moving viscous fluid, the attractive force between fluid particles and solid particles (adhesive forces) is greater than the force between fluid particles themselves (cohesive forces).

This force imbalance reduces the relative velocity of the fluid at the interface to zero. This is the no-slip or adherence condition, defined for viscous flows when the continuum-medium concept is valid.

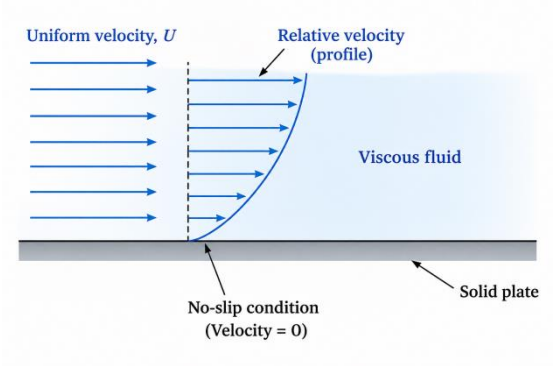


Figure 2. No-slip condition between a solid and a moving viscous fluid.

As with most approximations, the no-slip condition is not always valid in reality. For example, at very low pressure (e.g. at high altitude), even when the continuum approximation remains valid, there may be so few molecules near the surface that they "bounce" off the surface. For a fluid with zero viscosity (inviscid), there is no adhesion between the fluid and the solid, and there is slip at the fluid–solid interface.

2.2.2 Newton's Law

Consider the flow of a viscous fluid between two parallel plates: one fixed and the other moving at a certain velocity U_T (Fig. 3). When the moving plate starts moving, the initially stationary fluid is carried along by the motion. Very rapidly, the fluid reaches a constant velocity: the flow is steady (velocity independent of time). The fluid in contact with each plate moves at the same velocity as the plate (no-slip condition). Between the two plates, the fluid velocity therefore increases from zero at the lower plate to U_T at the upper moving plate. A shear stress $\tau = F/A$ must be applied to maintain the upper plate at constant velocity UT. This stress is related to fluid viscosity and the velocity gradient or shear rate $\partial u/\partial y$. Viscous fluids called "Newtonian" fluids obey the linear relationship given by Newton's law of viscosity. The proportionality factor is the dynamic viscosity, usually simply called viscosity, denoted by μ.

$$\tau = \mu \, (\partial u/\partial y) \quad \text{(I.3)}$$

The unit of shear stress τ is [N/m²] or [Pa], the unit of shear rate $\partial u/\partial y$ is [s^{-1}], and the unit of viscosity is [N·s/m²], [Pa·s], or [Poise].

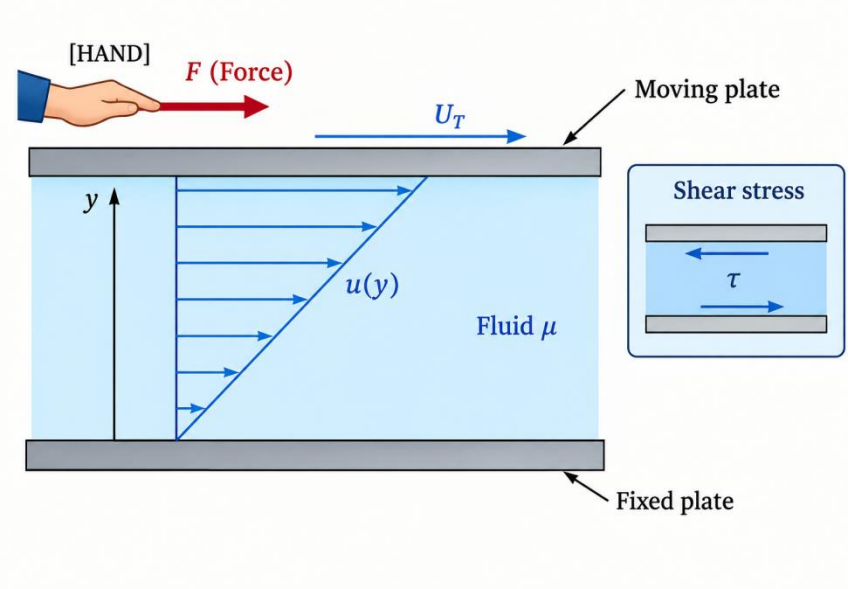


Figure 3. Couette flow.

In general, viscosity depends on the state of a fluid, such as its temperature, pressure, and rate of deformation. However, dependence on some of these properties is negligible in certain cases. A fluid with zero viscosity is called ideal or inviscid. Zero viscosity is observed only at very low temperatures in superfluids; otherwise, the second law of thermodynamics requires all fluids to have positive viscosity. In aerodynamics, viscosity is essential for understanding how fluid flows around moving objects, particularly aircraft wings. Viscosity affects vortex formation, drag, and dissipation of kinetic energy.

3. Types of Flows

3.1 Time Variation

According to temporal variation, flow can be classified into two types:

- Steady (permanent): $\partial/\partial t = 0$.
- Unsteady (non-permanent): $\partial/\partial t \neq 0$.

A steady flow is one in which parameters (velocity, pressure, etc.) may change from one point to another in space but do not change with time. In real conditions it is very rare to have flows with parameters absolutely constant in time. Parameters generally change with time, but their variation remains within a small range so that the average of a particular parameter remains constant over a fixed period.

If, at any point in space, the flow parameters change with time, the flow is called unsteady.

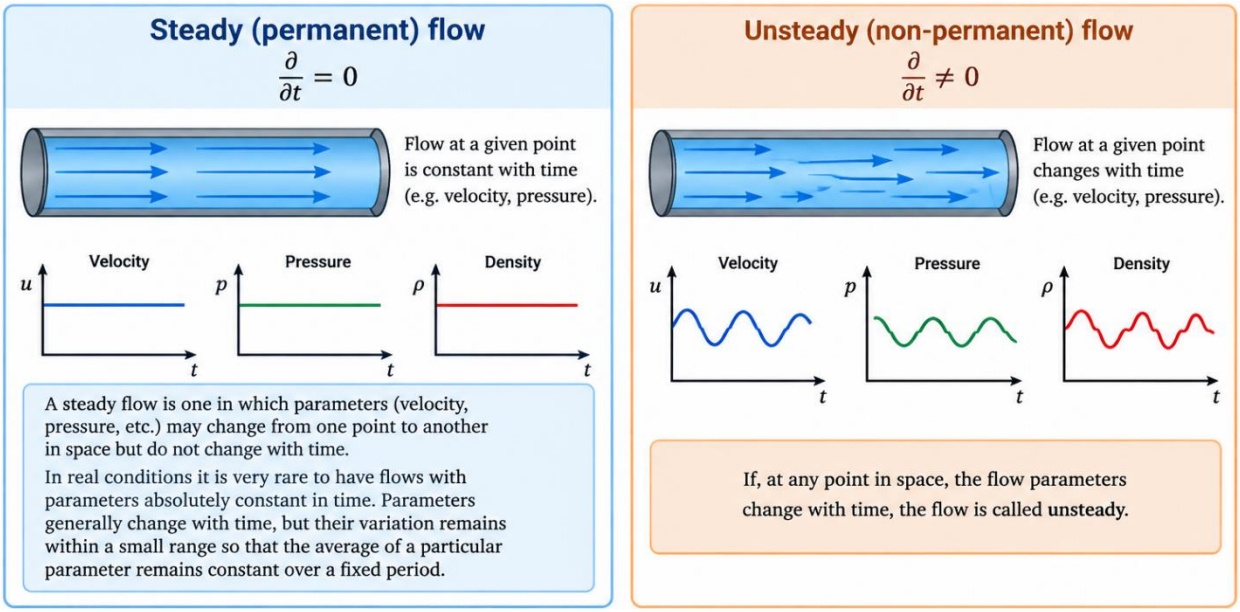


Figure 4. Types of flow according to temporal variation.

3.2 Spatial Variation

Although, in general, all fluids flow in a three-dimensional (3D) manner, varying in all directions in pressure, velocity, and other flow properties, in many cases the most important change occurs in only two directions (2D) or even one direction (1D). In such cases, changes in the other directions can effectively be ignored to simplify the analysis.

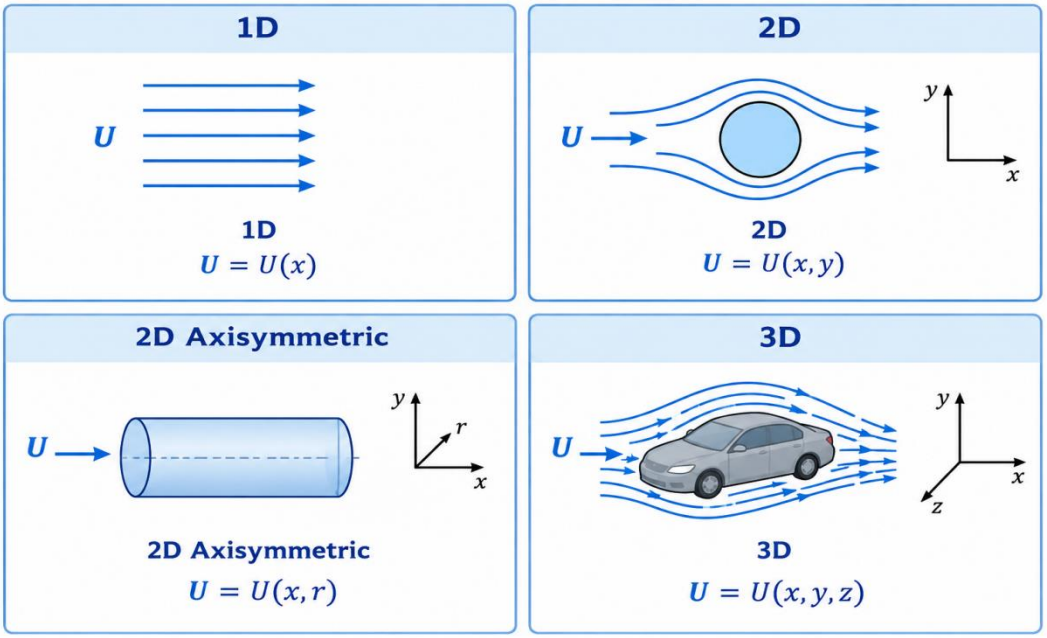


Figure 5. Types of flow according to spatial variation.

3.3 Viscous Effects

- Inviscid flow (perfect or ideal fluid) → Euler and Bernoulli equations.
- Viscous flow (real fluid) → Navier–Stokes equations.

A flow is assumed inviscid when viscosity is very small or almost negligible, i.e. there is no friction between fluid layers. The shear stress acting through the body is very small or effectively zero. Inviscid flows do not really exist in nature; however, there are many practical aerodynamic flows in which viscous effects are small, and the flow can be modeled as inviscid. Inviscid-flow problems are generally solved using standard aerodynamic methods based on Euler's equation.

A flow in which viscous forces are present is called a viscous flow. Friction forces act between fluid layers. In viscous flow, the velocity at the body surface is zero because of viscosity (no-slip condition).

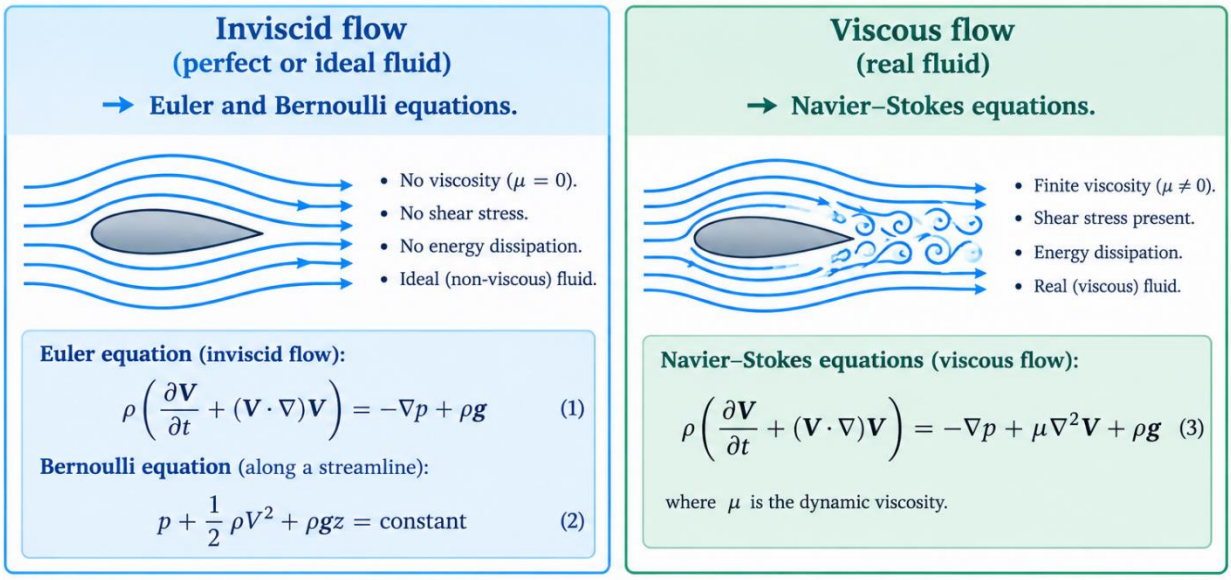


Figure 6. Type of flow according to viscosity.

Because viscosity makes problem solving much more complex, it is often convenient to divide real fluid flows into different regions or flow zones that can be considered mainly inviscid or mainly viscous. Such a zonal decomposition is often used to analyze aerodynamic problems, especially when viscous-flow regions are confined to small portions of the overall flow field. The critical part is how the regions are coupled computationally, i.e. how information is transmitted between zones. A zonal-flow simulation is more expensive than a completely inviscid solution, but it is generally much less expensive than a full viscous simulation.

3.3.1 Reynolds Number

The Reynolds number, denoted Re, is a dimensionless parameter used as a measure of the relative importance of viscous effects in a fluid. In aerodynamics, the Reynolds number is the ratio of inertial forces to viscous forces.

$$\mathbf{F_{viscous} = viscous\ stress \times area = \tau L^2 = \mu\left(\frac{U}{L}\right)L^2 = \mu UL} \quad \textbf{(I.4)}$$

$$\mathbf{F_{inertial} = mass \times acceleration = \rho L^3\left(\frac{U}{t}\right) = \rho L^2\left(\frac{L}{t}\right)U = \rho L^2 U^2} \quad \textbf{(I.5)}$$

$$\mathbf{Re = \frac{F_{inertial}}{F_{viscous}} = \frac{\rho U^2 L^2}{\mu UL} = \frac{\rho UL}{\mu}} \qquad \textbf{(I. 6)}$$

where L is the characteristic length of the object [m] and U is the flow (or object) velocity [m/s]. For inviscid flow the Reynolds number tends toward infinity because it is inversely proportional to viscosity, which tends toward zero. The Navier–Stokes equation can then be simplified to Euler's equation. For viscous flow the Reynolds number is finite. Practically all flows are viscous, regardless of the magnitude of the force, although the effect may be small.

3.4 Flow Regime

Fluid flows can be classified into two main types: laminar flow and turbulent flow.

- Laminar: streamlines are smooth and regular and a fluid element moves smoothly along a streamline.
- Turbulent: streamlines break up and a fluid element moves randomly, irregularly, and tortuously.

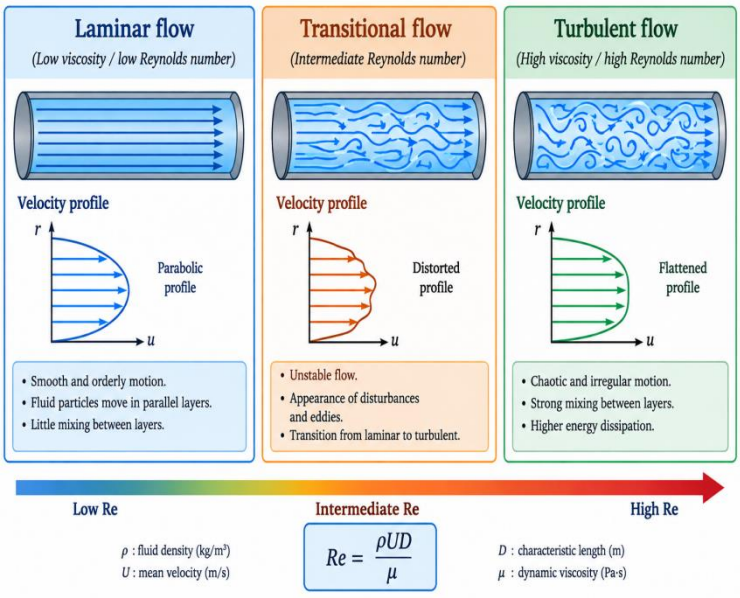


Figure 7. Flow type according to laminar or turbulent regime.

In laminar flow there is no mixing between fluid layers, which move smoothly as layers or sheets. Laminar flows, however, naturally become turbulent after some time (or downstream distance); such behavior is unavoidable, at least in a natural flow environment.

In turbulent flow, fluid layers mix. Fluid viscosity creates internal shear stresses that cause rotating fluid elements and the formation of vortices and turbulence. Turbulent flows are not easy to describe from a measurement or computational point of view, and statistical methods must be used. A turbulent flow is often described as stochastic or non-deterministic because of its random nature.

Turbulent flows are the most common flows encountered in practice, but laminar flows can exist under certain specific circumstances, at least for a short period after the initial formation of the flow. A third type may be called transitional flow: neither completely laminar nor completely turbulent. Laminar flows pass through this state before becoming fully turbulent; this is a process rather than a sudden event. At low Reynolds numbers, flows tend to be dominated by laminar motion, whereas at high Reynolds numbers they tend to be turbulent. The critical Reynolds number Rec is a specific Reynolds-number value from which the flow of a fluid around an object changes significantly. It is associated with laminar-to-turbulent transition. For a flat plate, for example, Rec ≈ 500,000. However, the critical Reynolds number can vary with many factors, such as object shape, surface roughness, and flow conditions.

3.5 Compressibility

- Incompressible flow → $\rho \approx$ constant (Ma < 0.3).
- Compressible flow → $\rho \neq$ constant (Ma > 0.3).

All flows are more or less compressible; a truly incompressible flow, in which density is exactly constant, does not occur in nature. However, as with inviscid flow, many aerodynamic problems can be modeled as incompressible without detrimental loss of accuracy. Homogeneous liquid flows, for example, are treated as incompressible, so most hydrodynamic problems assume ρ = constant. Air compressibility can be neglected for Mach numbers below 0.3 because the density change due to velocity is about 5% in this case. This flight regime (Ma < 0.3) was that of all aircraft from the Wright brothers' first flight in 1903 until shortly before the Second World War, and it remains the regime of most small general-aviation aircraft today. Consequently, there is a large body of experimental and theoretical aerodynamic data for incompressible flows.

Exercise:

Water flows through a circular pipe of diameter D=0.05 m with an average velocity V=2 m/s.
The density of water is ρ=1000 kg/m^3, and its dynamic viscosity is $\mu=1.0\times10^{-3}$ Pa.s.

Calculate the Reynolds number and determine the type of flow.

Solution

$Re = \frac{\rho VD}{\mu}, Re = \frac{1000\times2\times0.05}{1.0\times10^{-3}}, Re = 100000$, Since Re>4000, the flow is **turbulent**

3.6 Mach Number

The Mach number is a dimensionless number, denoted Ma, expressing the ratio of an object's speed in a fluid to the speed of sound in that fluid. The speed of sound in a gas varies with its nature and temperature, so Mach number does not correspond to a fixed speed; it depends on local conditions. At usual temperatures in air, the speed of sound is about 340 m/s or 1234 km/h. Mach number measures the ratio between motion-related forces and fluid compressibility.

$$\mathbf{a} = \sqrt{\boldsymbol{\gamma RT}} \quad \textbf{(I.7)}$$

where U is the object velocity relative to its environment [m/s] and a is the speed of propagation of sound in the considered environment [m/s]. For air treated as an ideal gas:

$$\boldsymbol{Ma} = \frac{\boldsymbol{U}}{\sqrt{\boldsymbol{\gamma RT}}} \quad \textbf{(I.8)}$$

- ✓ **subsonic** (Ma<0.8): The entire flow speed is below the speed of sound.
- ✓ **Transonic** (0.8<Ma<1.2): The flow transitions through the speed of sound, containing both subsonic and supersonic regions.
- ✓ **Supersonic** (1.2<Ma<5): The entire flow speed is faster than the speed of sound.
- ✓ **Hypersonic** (Ma>5): The flow speed is extremely fast, far exceeding the speed of sound, where high-temperature gas effects dominate.

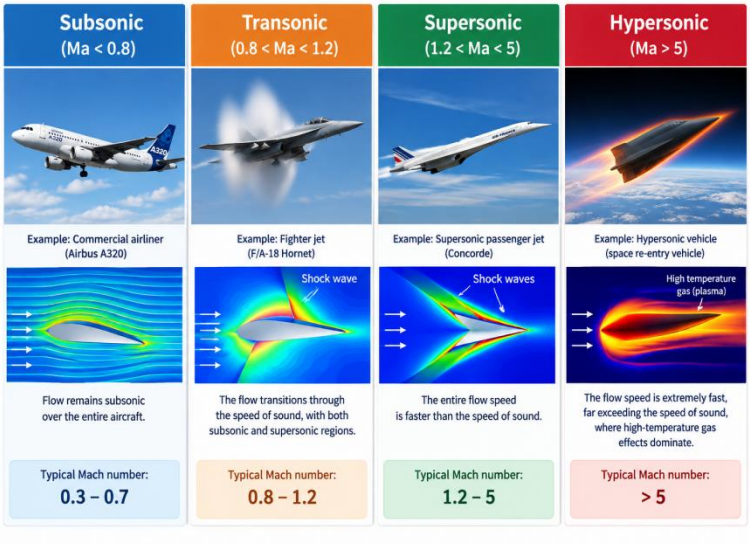


Figure 8. Examples of flow type according to Mach number.

Example Subsonic flow — Ma<0.8: a commercial passenger aircraft such as an Airbus A320.

- U<a
- Pressure disturbances can propagate upstream.
- The flow around the aircraft remains essentially subsonic.
- No strong shock waves are normally present.

4. Aerodynamic Forces

An aerodynamic force is a force exerted by air on a solid body moving relative to the air. Two forces give rise to the aerodynamic force: the force due to pressure on the body surface and the force due to shear stress caused by air viscosity. Pressure and shear stress are both given as a ratio between an applied force and an area. They have the same dimensions and units, [N/m²] or [Pa]. However, pressure always acts perpendicular to the surface (normal force), whereas shear stress acts along the surface plane (parallel or tangential force).



Figure 9:Aerodynamic forces.

$$p = \frac{F_{\perp}}{S}, \qquad \tau = \frac{F_{\parallel}}{S} \qquad (I.7)$$

where:

- p= **pressure** (normal force per unit area)
- τ = **shear stress** (tangential force per unit area)
- F⊥ = force **perpendicular** to the surface
- F∥ = force **parallel** to the surface
- S = surface area

For an airfoil, the pressure distribution acts locally perpendicular (normally) to the airfoil surface, while the shear distribution acts locally parallel to the surface. Pressure and shear-stress distributions are not uniform and can be positive or negative, e.g. higher pressure pushing inward toward the surface or lower pressure pulling away from it. Boundary-layer-induced shear stresses can also be positive (flow moving downstream) or negative (e.g. reversed flow). The detailed distribution of shear stress and pressure over the body surface is difficult to obtain. Often only the integrated or resultant effects are needed.

Adding the local pressure contribution at every point along the surface (integration) gives the net pressure force acting on the airfoil. Similarly, adding the shear contribution along the airfoil surface gives a net shear force.

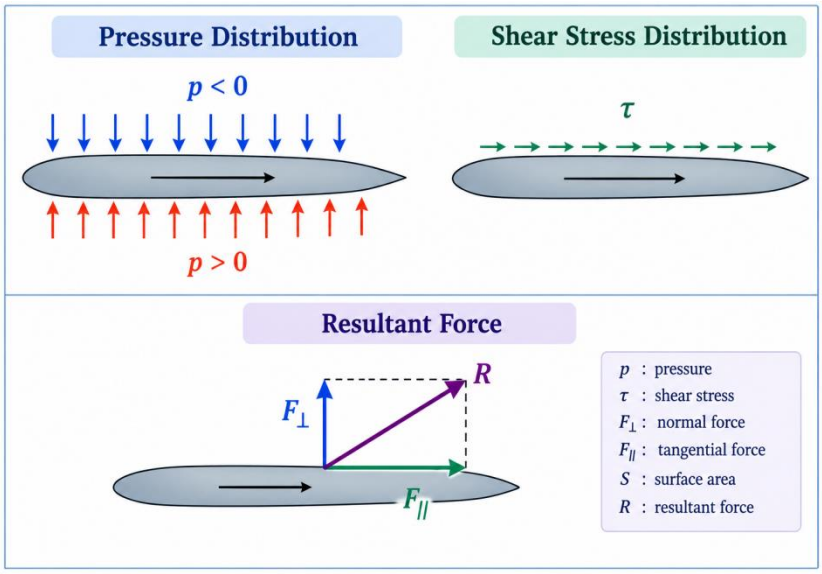


Figure 10. Distribution of pressure (normal) and shear (parallel) forces on an airfoil and their resultant.

$$\boldsymbol{R = \int_S p\ dS + \int_S \tau\ dS} \qquad \boldsymbol{(I.8)}$$

The resultant force R can be divided into two forces:

- F_L: lift, perpendicular to the direction of relative motion.
- F_D: drag, parallel to the direction of relative motion.

$$\boldsymbol{R = F_L + F_D} \qquad \boldsymbol{(I.9)}$$

5. Continuity Equation

5.1. Fluid particle:

A fluid particle is chosen as an elementary entity that allows a complete description of flows:

It is a “packet” of molecules surrounding a given point M; these molecules are assumed to have the same velocity at the same instant.

5.2.Eulerian and Lagrangian Descriptions:

5.2.1.Eulerian Description:

This description of the flow consists of determining, at a given instant **t,** the set of velocities associated with each point in the space occupied by the fluid:

The velocity $\rightarrow V_M(t)$ associated with point M is evaluated over time.

At each instant t, the fluid flow is described by means of a velocity vector field.

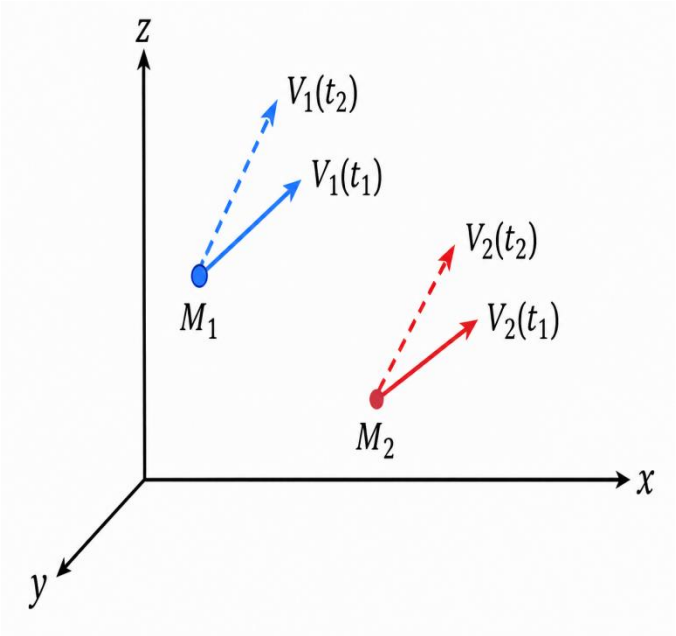


Figure 11.Eulerian description.

In this Eulerian description, a "streamline" is the curve that is tangent to the velocity vectors at every point.

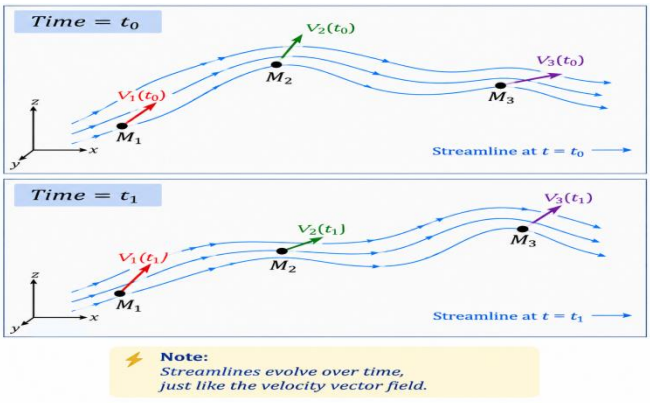


*Figure 12.*streamline.

5.2.2.Lagrangian Description:

This description of the flow consists of following a given fluid particle during its motion through the fluid.

Here, the positions of the particles are used to describe the flow.

Thus, the geometric locus of the successive positions occupied by a particle is called the "trajectory" of that particle.

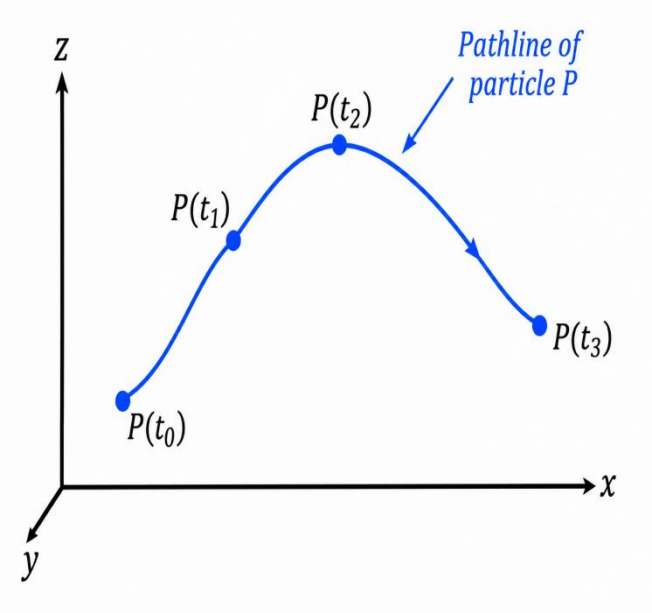


Figure 13.Lagrangian Description.

- Streamlines and trajectories must not be confused. They are two different concepts.

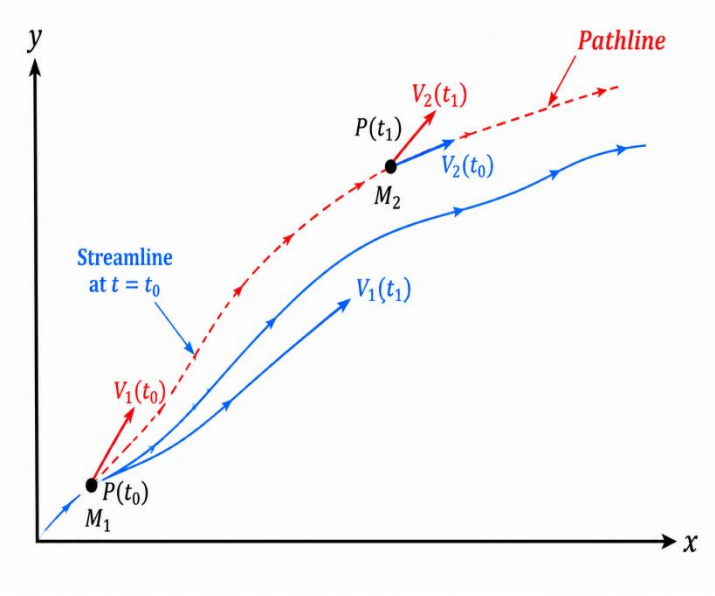


Figure 14.Comparison Between a Pathline and a Streamline

Remark:

If the flow is steady, the velocity vector field is constant in time; streamlines and trajectories coincide.

5.2.2.1.Streakline:

All particles that have passed through the same point E are located, at time t, on a curve called a "streakline".

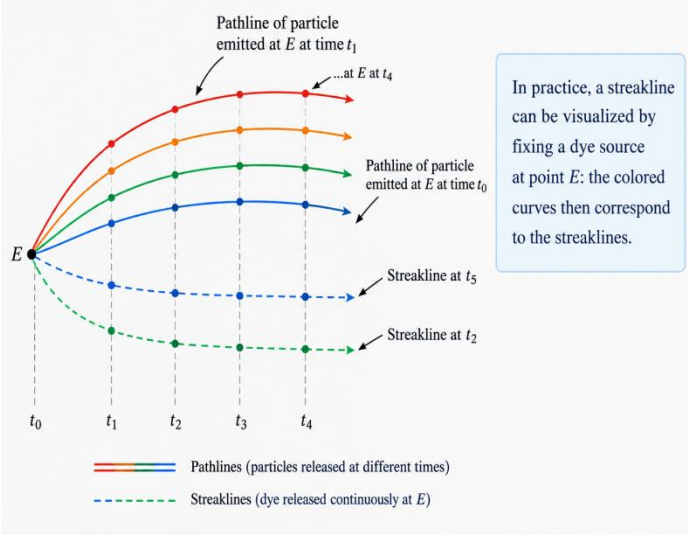


Figure 15:Illustration of Pathlines and Streaklines in Fluid Flow.

5.2.2.2.Steady flow:

A flow is said to be steady (or stationary) when the velocity vector field is constant; it does not vary with time.

In this case:

- the streamlines are fixed in space.
- the trajectories coincide with the streamlines.
- the streaklines also coincide with the streamline

Streamline ≡ trajectory ≡ streakline

Flow rate:

Through surface S, the mass flow rate of the fluid

is given by:

$$\dot{m} = \int_S \rho \vec{V} \cdot \vec{n}\, dS \qquad \textbf{(I-10)}$$

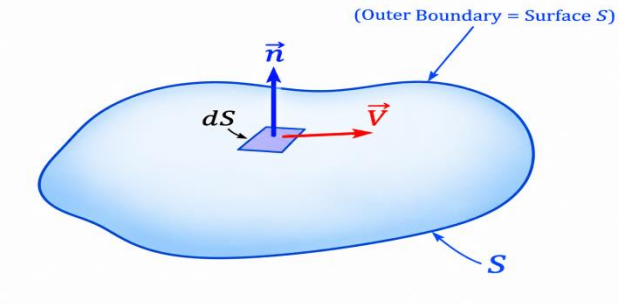


Figure 16:Mass Flux Through an Elemental Surface dS.

Through surface S, the volumetric flow rate of the fluid is given by:

$$Q = \int_S \vec{V} \cdot \vec{n}\, dS \quad \textbf{(I.11)}$$

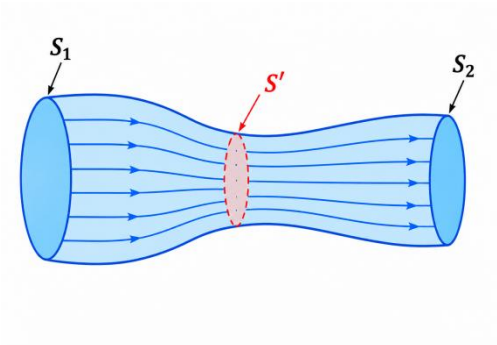


Figure 17.Mass Conservation Between Two Cross-Sections.

All streamlines based on the same closed curve form a surface (S′) called a "streamtube".

If the flow is steady (the tube does not change with time), the mass flow rate is conserved:

$$\dot{m}_{S_1} = \dot{m}_{S_2} \quad \textbf{(I.12)}$$

where $\dot{m}$ is the **mass flow rate**, ρ the density, V the velocity, and S the cross-sectional area ($\dot{m}=\rho VS$)

* If the fluid is incompressible, the volumetric flow rate is conserved:

$$Q = Q_1 = Q_2 = V_1 S_1 = V_2 S_2 \quad \textbf{(I.13)}$$

Therefore, the **continuity equation** is:

$$V_1 S_1 = V_2 S_2 \quad \textbf{(I.14)}$$

5.3.Derivation of the Continuity Equation

The continuity equation expresses the **principle of conservation of mass**.

The change in mass during a time interval (dt) within a fluid volume element must be equal to the mass of fluid entering the element minus the mass of fluid leaving it.

We consider a differential fluid volume element: [dV=dx, dy, dz]

The fluid velocity is expressed as:

$$\vec{V} = V_x \vec{e}_x + V_y \vec{e}_y + V_z \vec{e}_z \quad \textbf{(I.15)}$$

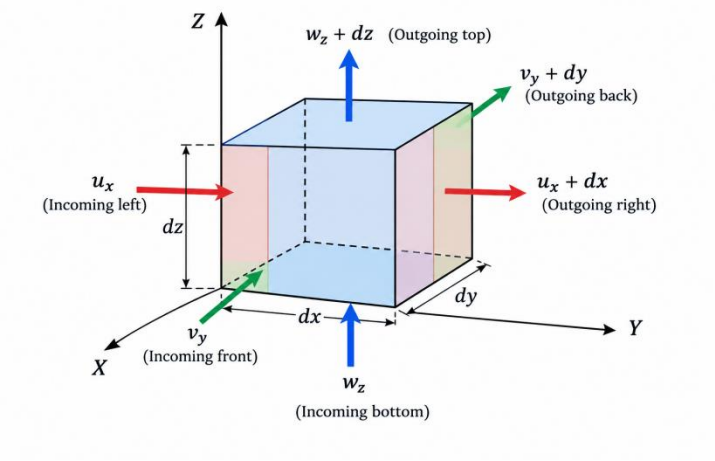


Figure 18.Continuity Equation.

1. Balance along the y-axis

Consider the two faces perpendicular to the **y-axis**..

The mass entering during **dt** is:

$$dm_{entrant} = \rho . Vy . dx dz dt \qquad \textbf{(I.16)}$$

The mass leaving is:

$$dm_{sortant} = \left[\rho V_y + \frac{\partial(\rho V_y)}{\partial y} dy\right] dx\, dz\, dt \qquad \textbf{(I.17)}$$

The balance along the y-axis gives:

$$dm_y = \left[\rho V_y - \left(\rho V_y + \frac{\partial(\rho V_y)}{\partial y} dy\right)\right] dx\, dz\, dt \qquad \textbf{(I.18)}$$

Therefore:

$$dm_y = -\frac{\partial(\rho V_y)}{\partial y} dx\, dy\, dz\, dt \qquad \textbf{(I.19)}$$

Similarly, along the x- and z-axes:

$$dm_x = -\frac{\partial(\rho V_x)}{\partial x} dx\, dy\, dz\, dt \qquad \textbf{(I.20)}$$

$$dm_z = -\frac{\partial(\rho V_z)}{\partial z}\, dx\, dy\, dz\, dt \qquad \textbf{(I.21)}$$

The mass of this fluid element can be expressed as:

$$m = \rho\, dx\, dy\, dz \qquad \textbf{(I.22)}$$

During the time interval (dt), the rate of change of mass is :

$$\frac{dm}{dt} = \frac{\partial \rho}{\partial t}\, dx\, dy\, dz \qquad \textbf{(I.23)}$$

Therefore, the change in mass during (dt) is:

$$dm = \frac{\partial \rho}{\partial t}\, dx\, dy\, dz\, dt \quad \textbf{(I.24)}$$

According to the principle of conservation of mass, this change must be equal to the **mass flow entering the volume element minus the mass flow leaving it**.

2. Mass Balance Through the Six Faces

The total change in mass due to the fluid fluxes through the six faces is therefore:

$$dm_{flux} = -\left[\frac{\partial(\rho V_x)}{\partial x} + \frac{\partial(\rho V_y)}{\partial y} + \frac{\partial(\rho V_z)}{\partial z}\right] dx\, dy\, dz\, dt \qquad \textbf{(I.25)}$$

For the volume element:

$$dV = dx\, dy\, dz$$

And the compact vector form:

$$dm_{flux} = -\nabla \cdot (\rho \vec{V})\, dV\, dt \quad \textbf{(I.26)}$$

The net mass flow rate into the differential volume is:

$$\nabla \cdot (\rho \vec{V})\, dV \quad \textbf{(I.27)}$$

Therefore:

$$\frac{\partial \rho}{\partial t}\, dV = -\nabla \cdot (\rho \vec{V})\, dV \quad \textbf{(I.28)}$$

Since

$$dV \neq 0$$

we obtain the **general continuity equation**:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0 \quad \textbf{(I.29)}$$

For an **incompressible fluid**, where ρ is constant:

$$\nabla \cdot \vec{V} = 0$$

This is the **differential form of the continuity equation for incompressible flow**.

In Cartesian Coordinates

The continuity equation for an incompressible and conservative flow, expressed in Cartesian coordinates, is:

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0 \quad \textbf{(I.30)}$$

Example

A.If the velocity potential of an incompressible, two-dimensional fluid flow is given by:

$$\phi(x,y) = -3xy^3 - x^2 + 3x^3y + f(y)$$

1.The function f(y) so that the continuity equation is satisfied.
2.The stream function.

B. The velocity components of an incompressible fluid flow are:

$$u = 2x^2y, \quad v = -2xy^2$$

1.The equations of the pathline described by the fluid particle.

Solution

Part A

1. Determine f(y) for continuity:

For an incompressible, 2D flow, the continuity equation is:

Continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

Velocity components from the velocity potential

$$u=\frac{\partial \phi}{\partial x}$$

$$v=\frac{\partial \phi}{\partial y}$$

Calculate u

$$u=\frac{\partial \phi}{\partial x}=\frac{\partial}{\partial x}\left(-xy^3-x^2+x^3y+f(y)\right)=-y^3-2x+3x^2y$$

Calculate v

$$v=\frac{\partial \phi}{\partial y}=\frac{\partial}{\partial y}\left(-xy^3-x^2+x^3y+f(y)\right)=-3xy^2+x^3+f'(y)$$

Now, plug these into the continuity equation:

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left(-y^3-2x+3x^2y\right)=-2+6xy$$

$$\frac{\partial v}{\partial y}=\frac{\partial}{\partial y}\left(-3xy^2+x^3+f'(y)\right)=-6xy+f''(y)$$

Set the sum to zero:

$$(-2+6xy)+(-6xy+f''(y))=0$$

Therefore:

$$-2+f''(y)=0 \quad \square \quad f''(y)=2$$

Integrating twice

$$f'(y)=\int 2\ dy=2y+C_1$$

$$f(y)=\int (2y+C_1)\ dy=y^2+C_1y+C_2$$

Since C1 and C2 do not affect the velocity field, we can choose:

$$f(y)=y^2$$

Thus:

$$u=-y^3-2x+3x^2y$$

$$v=-3xy^2+x^3+2y$$

2. Determine the stream function ψ(x,y)

The stream function is defined by:

$$u=\frac{\partial\psi}{\partial y}$$

$$v=-\frac{\partial\psi}{\partial x}$$

Integrating u with respect to y:

$$\psi=\int(-y^3-2x+3x^2y)\,dy$$

Therefore

$$\psi=-\frac{y^4}{4}-2xy+\frac{3}{2}x^2y^2+g(x)$$

Differentiate with respect to xx:

$$\frac{\partial\psi}{\partial x}=-2y+3xy^2+g'(x)$$

Since

$$-v=3xy^2-x^3-2y$$

we obtain:$-2y+3xy^2+g'(x)=3xy^2-x^3-2y$

Thus:$g'(x)=-x^3$

And

$$g(x)=-\frac{x^4}{4}$$

Hence, the stream function is

$$\boxed{\psi(x,y)=-\frac{y^4}{4}-2xy+\frac{3}{2}x^2y^2-\frac{x^4}{4}}$$

Part B — Pathline

For the **pathline**, you give a different velocity field:

$$u=2x^2y$$

$$v=-2xy^2$$

The differential equations are:

$$\frac{dx}{dt}=2x^2y$$

$$\frac{dy}{dt}=-2xy^2$$

Using

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$$

we obtain: $\frac{dy}{dx}=\frac{-2xy^2}{2x^2y}=-\frac{y}{x}$

Separate the variables:

$$\frac{dy}{y}=-\frac{dx}{x}$$

Integrating: $\ln|y|=-\ln|x|+C$

Therefore: $\ln|xy|=C$

and hence: $xy=C_1$

Or $y=\frac{C_1}{x}$

Position as a function of time

Substitute y=C1/x into : $\frac{dx}{dt}=2x^2y$

$$\frac{dx}{dt}=2x^2\frac{C_1}{x}=2C_1x$$

Therefore

$$\frac{dx}{x}=2C_1dt$$

Integrating:

$$\ln|x|=2C_1t+C_2$$

Thus: $x(t)=C_3e^{2C_1t}$

Since xy=C1

$$y(t)=\frac{C_1}{C_3}e^{-2C_1t}$$

So the pathline can be written as: $\boxed{x(t)=C_3e^{2C_1t}}$

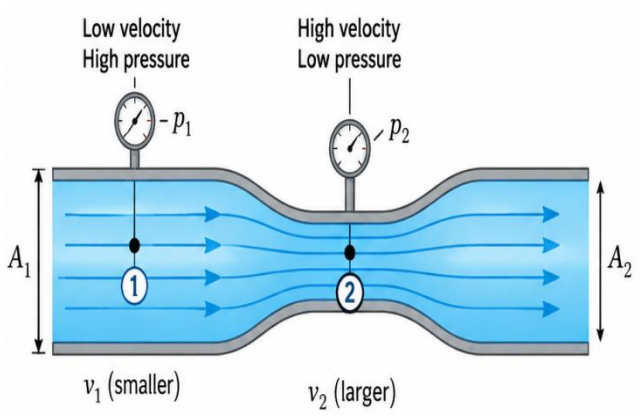
$$y(t)=\frac{C_1}{C_3}e^{-2C_1t}$$

Note: C1,C3,C4 are constants determined by the initial position of the particle at t=0.

6. Bernoulli Equation

Consider an **incompressible**, **inviscid** fluid undergoing **steady flow** through a pipe.

We consider a fluid element moving between two sections, 1 and 2.



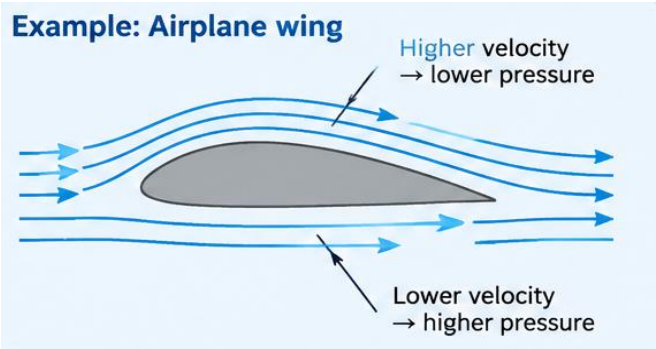


Figure 19.Bernoulli Equation.

This diagram illustrates a parcel of fluid moving through a pipe that changes in both **height** (from y_1 to y_2) and **cross-sectional area** (from A_1 to A_2). The goal is to calculate the total energy change of this fluid parcel to derive Bernoulli's principle.

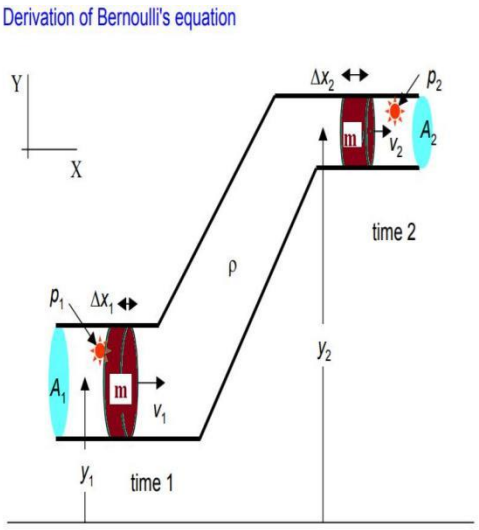


Figure 20.Bernoulli's Equation Derivation

At section 1:

- Pressure: p_1
- Velocity: V_1
- Elevation: z_1

At section 2:

- Pressure: p_2
- Velocity: V_2
- Elevation: z_2

1. Work of Pressure Forces

The pressure at the bottom pushes the fluid forward, doing positive work. The pressure at the top pushes back, doing negative work.On a section S1, the fluid exerts a pressure p1.

The elementary work is:

$$W_1 = p_1 S_1 dx_1 = p_1 dV \quad \textbf{(I.31)}$$

$$W_2 = -p_2 dV \quad \textbf{(I.32)}$$

Total work:

$$W = p_1 dV - p_2 dV = (p_1 - p_2) dV \quad \textbf{(I.33)}$$

You can also write the last equation directly as: $\boxed{W = (p_1 - p_2) dV}$

2. Variation of Kinetic Energy

The mass of fluid:

$$\boldsymbol{dm = \rho dV} \quad \textbf{(I.34)}$$

The kinetic energy:

$$\boldsymbol{dE_c = \frac{1}{2}\rho dV(V_2^2 - V_1^2)} \quad \textbf{(I.35)}$$

3. Variation of Potential Energy

The gravitational potential energy:

$$\boldsymbol{E_p = mgz} \quad \textbf{(I.36)}$$

Therefore, the variation:

$$\boldsymbol{dE_p = \rho g dV(z_2 - z_1)} \quad \textbf{(I.37)}$$

You now have all three components needed to apply the **Conservation of Energy** to a fluid element :

Work done by pressure forces: $dE_c = \frac{1}{2}\rho dV(V_2^2 - V_1^2)$

Change in kinetic energy: $dE_c = \frac{1}{2}\rho dV(V_2^2 - V_1^2)$

Change in potential energy: $dE_p = \rho g dV(z_2 - z_1)$

According to the principle of conservation of energy, the **net work done on the fluid equals the change in its total mechanical energy**:

$$\mathbf{W=dEc+dEp}$$

If you substitute the three equations above into this statement and divide everything by dV, you will get:

$$\boldsymbol{p_1 - p_2 = \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(z_2 - z_1)} \quad \textbf{(I.38)}$$

Bernoulli's equation

$$\boldsymbol{p_1 + \frac{1}{2}\rho v_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g z_2} \quad \textbf{(I.39)}$$

General form

$$p + \frac{1}{2}\rho v^2 + \rho g z = constant \quad \textbf{(I.40)}$$